MAT

Serie A

Conferencias, seminarios y trabajos de Matemática

ISSN:1515-4904

5

VI Seminario sobre Problemas de Frontera Libre y sus Aplicaciones.

Tercera Parte

Departamento de Matemática, Rosario, Argentina
2001

UNIVERSIDAD AUSTRAL
FACULTAD DE CIENCIAS EMPRESARIALES

MAT

Serie A: CONFERENCIAS, SEMINARIOS Y TRABAJOS DE MATEMATICA

ISSN: 1515-4904
Propiedad de ACES

MAT es una publicación del Departamento de Matemática de la Facultad de Ciencias Empresariales de la Universidad Austral (FCE-UA) cuyo objetivo es contribuir a la difusión de conocimientos y resultados matemáticos. Se compone de dos series:

- Serie A: CONFERENCIAS, SEMINARIOS Y TRABAJOS DE MATEMATICA.
- Serie B: CURSOS Y SEMINARIOS PARA EDUCACION MATEMATICA.

La Serie A contiene trabajos originales de investigación y/o recapitulación que presenten una exposición interesante y actualizada de algunos aspectos de la Matemática, además de cursos, conferencias, seminarios y congresos realizados en el Depto. de Matemática. El Director, los miembros del Comité Editorial y Científico y/o los árbitros que ellos designen serán los encargados de dictaminar sobre los merecimientos de los artículos que se publiquen.

La Serie B se compone de cursos especialmente diseñados para profesores de Matemática de cada uno de los niveles de educación: E.G.B., Polimodal, Terciaria y Universitaria.

Además, se publican bajo el título **MAT-** PREPUBLICACIONES DE MATEMATICA, versiones preliminares de trabajos inéditos de investigación de los integrantes del Departamento y colaboradores.

La serie A y las Prepublicaciones podrán ser consultadas en: www.austral.edu.ar/MAT

ISSN 1515-4904

MAT

SERIE A : CONFERENCIAS, SEMINARIOS Y TRABAJOS DE MATEMÁTICA

No. 5

VI SEMINARIO SOBRE PROBLEMAS DE
FRONTERA LIBRE Y SUS APLICACIONES
Tercera Parte

Domingo A. Tarzia (Ed.)

INDICE

Rosario, Octubre 2001

El VI Seminario sobre Problemas de Frontera Libre y sus Aplicaciones tuvo lugar en el Departamento de Matemática de la FCE de la Universidad Austral, en Rosario, del 16 al 18 de Diciembre de 1998. Fue realizado con el apoyo del Proyecto de Investigación Plurianual del CONICET "Problemas de Frontera Libre para la Ecuación del Calor-Difusión" y gracias a un subsidio otorgado por el CONICET.

Los problemas de frontera libre son aquellos problemas de contorno donde interviene además, una superficie incógnita (la "frontera libre") que separa dos o más regiones, sobre la cual se conocen datos que dependen del modelo analizado. Según el número de dimensiones del espacio, en lugar de una superficie de separación se podrá tener una curva o un número finito de puntos.

El avance considerable que se ha obtenido en el desarrollo teórico de estos temas a nivel nacional y sus potenciales aplicaciones a la industria (electropintura, envenenamiento y regeneración de catalizadores; combustión de sólidos; solidificación de aleaciones binarias; soldadura de metales; colada contínua del acero; congelación de alimentos en la industria frigorífica; almacenamiento de energía térmica de origen solar por cambio de fase; oxidación del zirconio y fusión del dióxido de uranio en reactores nucleares, en caso de accidentes; procesos de ablación térmica; difusión-consumo de oxígeno en tejidos vivos, para el tratamiento médico de tumores mediante la aplicación de radiaciones; problemas de control óptimo ligados a procesos con cambio de fase; solidificación de suelos húmedos; derretimiento de glaciares; crecimiento de raíces de cultivo; precio en las opciones americanas; etc.) impulsaron su realización, prosiguiendo la línea de los Seminarios anteriores, con el objetivo de facilitar la interacción entre las personas y grupos de investigación que trabajan en dichos problemas y en temas conexos, y de despertar el interés y promover el acercamiento de jóvenes graduados.

COMITE ORGANIZADOR: Dr. L. A. Caffarelli, Austin - Texas (EEUU); Dr. E. A. García, Buenos Aires; Dr. D. A. Tarzia, Rosario (Coordinador); Ing. L. T. Villa, Salta; Dra. N. Wolanski, Buenos Aires.

SECRETARÍA LOCAL: Lic. A. C. Briozzo, Lic. G. G. Garguichevich, Ariel Lombardi, Lic. P. R. Marangunic, Lic. M. F. Natale, Lic. E. A. Santillan Marcus (Coordinador), Lic. M. C. Sanziel

Esta tercera parte contiene cinco de las conferencias y comunicaciones presentadas. La nómina general se incluye, en orden cronológico, en MAT – Serie A, 3 (2001), pág. 42.

Los manuscritos fueron recibidos y aceptados en agosto de 2001.

On A Two-Phase Stefan Problem with Nonlinear Thermal Coefficients *

Adriana C. BRIOZZO (1) **and Domingo A. TARZIA** (1) (2)

(1) Depto. Matemática, FCE,Universidad Austral,
Paraguay 1950, S2000FZF Rosario, ARGENTINA.
(2) CONICET, ARGENTINA.
E-mail: Adriana.Briozzo@fce.austral.edu.ar;
Domingo.Tarzia@fce.austral.edu.ar

Abstract

We review some recent results concerning a heat conduction problem with a particular nonlinear thermal coefficients in both solid and liquid phases for a semi-infinite material $x > 0$, with phase change temperature T_1, an initial temperature $T_2\,(> T_1)$ and a heat flux of the type $q(t) = \frac{q_0}{\sqrt{t}}$ imposed on the fixed face $x = 0$.

We determine necessary and/or sufficient conditions on the parameters of the problem in order to obtain an instantaneous nonlinear two-phase Stefan problem (solidification process).We also give the corresponding explicit solution.

Resumen: Se da una revisión de algunos resultados recientes que conciernen a una ecuación del calor con particulares coeficientes térmicos no lineales en ambas fases sólida y líquida de un cuerpo semi-infinito $x > 0$, con una temperatura de cambio de fase T_1, una temperatura inicial $T_2\,(> T_1)$ y un flujo de calor del tipo $q(t) = \frac{q_0}{\sqrt{t}}$ que se impone en el borde fijo $x = 0$. Se determinan condiciones necesarias y suficientes sobre los parámetros del problema con el objetivo de obtener un problema no lineal instántaneo de Stefan a dos fases (proceso de solidificación). Se da también la correspondiente solución explícita.

Key words: Stefan problem, Instantaneous phase-change problem, Free boundary problem, Nonlinear thermal coefficients, Explicit solution.

Palabras claves: Problema de Stefan, Problema de cambio de fase instántaneo, Problemas de frontera libre, Coeficientes térmicos no lineales, Solución explícita.

AMS Subject classification: 35R35, 80A22, 35C05.

1. INTRODUCTION

We consider the two-phase Stefan problem (solidification process)[Ta2] with nonlinear thermal coefficients for a semi-infinite region $x > 0$ with phase change temperature T_1 , an

*MAT - Serie A, 5 (2001), 1-10.

initial temperature $T_2 > T_1$ and an imposed heat flux of the type $q(t) = \frac{q_0}{\sqrt{t}}$, $(q_0 > 0)$ on the fixed face $x = 0$. For $t > 0$ we are going to determine, if there exist, the temperature distribution $u(x,t)$ and the free boundary $x = y(t)$, where

$$u(x,t) = \begin{cases} u_1(x,t) < T_1 & 0 < x < y(t) \\ T_1 & x = y(t) \\ u_2(x,t) > T_1 & x > y(t) \end{cases} , \tag{1.1}$$

which must verify the following conditions

$$C_1(u_1)\frac{\partial u_1}{\partial t} = \frac{\partial}{\partial x}\left[K_1(u_1)\frac{\partial u_1}{\partial x}\right] \quad , \quad 0 < x < y(t) \quad , \quad t > 0 \; , \tag{1.2}$$

$$C_2(u_2)\frac{\partial u_2}{\partial t} = \frac{\partial}{\partial x}\left[K_2(u_2)\frac{\partial u_2}{\partial x}\right] \quad , \quad x > y(t) \quad , \quad t > 0 \; , \tag{1.3}$$

$$y(0) = 0 \tag{1.4}$$

$$u_2(x,0) = T_2 > T_1 \quad , \quad x > 0 \; , \tag{1.5}$$

$$u_1(y(t),t) = u_2(y(t),t) = T_1 \quad , \quad t > 0 \; , \tag{1.6}$$

$$K_1(u_1)\frac{\partial u_1}{\partial x} - K_2(u_2)\frac{\partial u_2}{\partial x} = Ly'(t) \quad , \quad on\; x = y(t) \quad , \quad t > 0 \; , \tag{1.7}$$

$$K_1(u_1(0,t))\frac{\partial u_1}{\partial x}(0,t) = \frac{q_0}{\sqrt{t}} \quad , \quad t > 0 \; , \tag{1.8}$$

where

x : spatial coordinate, t : time,
$u_i(x,t)$: temperature distribution for phase i ,
T_1 : phase-change or freezing temperature,
T_2 : initial temperature, L : volumetric latent heat ,
$C_i(u_i)$: volumetric heat capacity for phase i,
$K_i(u_i)$: thermal conductivity for phase i ,
$y(t)$: free boundary (solid-liquid interface) at time t,
q_0 : positive given constant which characterizes the heat flux on the fixed face,
$i = 1$: solid phase , $i = 2$: liquid phase .

We assume that the volumetric heat capacity and the thermal conductivity for each phase i $(i = 1, 2)$ are relationed as follow :

$$C_i(u_i) = \frac{K_i(u_i)c_0}{k_0 a_i^2\left[b_i - \frac{1}{k_0}\int_0^{\frac{u_i - T_1}{T_2 - T_1}} K_i\left(T_1 + (T_2 - T_1)z\right)dz\right]^2} \tag{1.9}$$

with the assumption given by

$$\frac{1}{k_0\,(T_2 - T_1)}\int_{T_1}^{T_2} K_2\,(z)\,dz < b_2 \tag{1.10}$$

where a_i, b_i $(i = 1,2)$ are positive constants and k_0, c_0 are scales for the thermal conductivity and volumetric heat capacity respectively. The heat flux condition of the type (1.8) was firstly considered in [Ta1] where an inequality for the coefficient q_0 was found in order to have an instantaneous two-phase Stefan problem with constant thermal coefficients, for both solid and liquid phases. Other problems in this direction are given by [BrTa1, HiHa, NaTa, Ro1, Ro2, SoWiAl]. The nonlinear relations (1.9) follows from the solidification of iron on a cooper base [TrBr]. Furthermore, these relations imply that the material is of the Storm's type, that is to say [BrNaTa, HiHa, NaTa, Ro2, St]

$$\frac{1}{\sqrt{K_i(u_i)C_i(u_i)}}\frac{d}{dT}\left(\log\sqrt{\frac{C_i(u_i)}{K_i(u_i)}}\right) = \frac{a_i}{\sqrt{c_0 k_0}(T_2 - T_1)} = const. \ , \ i = 1,2$$

The goal of this paper is to determine which conditions on the parameters of the problem (in particular q_0) must be satisfied in order to have an instantaneous phase-change process.

In Section 2 we consider the associated nonlinear heat conduction problem corresponding to the initial liquid temperature T_2 and the heat flux condition on $x = 0$ of the type $\frac{q_0}{\sqrt{t}}$ for $t > 0$. The nonlinear condition between the thermal conductivity heat capacity is supposed to be of the type (1.9). We give a necessary condition for the heat flux input coefficient q_0, *i.e.*

$$q_0 > \frac{\sqrt{c_0 k_0}(T_2 - T_1)}{a_2} Q^{-1}\left((k_0 b_2 (T_2 - T_1))^{-1}\int_{T_1}^{T_2} K_2(z)\,dz\right) \tag{1.11}$$

in order to obtain an instantaneous change phase process, where Q is the real function defined by

$$Q(x) = \sqrt{\pi}x\exp(x^2)(1 - \mathrm{erf}(x)) \ , \ x > 0, \tag{1.12}$$

with the properties $Q(0) = 0$, $Q(+\infty) = 1$, $Q'(x) > 0$, $\forall x > 0$, where the error function is given by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-w^2)dw \ . \tag{1.13}$$

In the Section 3 we consider the nonlinear two phase Stefan problem (1.2) – (1.8) and we prove that it admits a similarity solution if the condition (1.11) for the coefficient q_0 obtained in Section 2 is verified. Some of these results are proved in [BrTa2].

2.- A NONLINEAR HEAT CONDUCTION PROBLEM AND ITS INSTANTANEOUS PHASE-CHANGE PROCESS.

We consider a semi-infinite slab $x \geq 0$ of a material that freezes at temperature T_1. We suppose that it is initially hot at the uniform temperature $T_2 > T_1$ and it has nonlinear heat transfer coefficients. However, what happens if a heat flux of the type $\frac{q_0}{\sqrt{t}}$ is imposed at $x = 0$? Our interest is found relations among data in order to obtain an instantaneous phase change process, that is the temperature of the material at $x = 0$ must be less than T_1 for all positive time. Then, we consider the following nonlinear heat conduction problem corresponding to the initial phase (liquid phase) given by

$$C_2(u)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[K_2(u)\frac{\partial u}{\partial x}\right] \quad , \quad x > 0 \ , \quad t > 0 \ , \tag{2.1}$$

$$u(x,0) = T_2, \quad x > 0 \ , \tag{2.2}$$

$$K_2(u(0,t))\frac{\partial u}{\partial x}(0,t) = \frac{q_0}{\sqrt{t}} \ , \quad t > 0 \ , \tag{2.3}$$

where K_2 and C_2 have the relation (1.9).

Then the question that follows is: Which conditions must be satisfied for the parameters q_0, T_1, T_2, K_2 and C_2 in order to have that the temperature $u(0,t) < T_1, \ \forall \, t > 0$? If the answer is affirmative we can assure [SoWiAl, Ta1, TaTu] that the phase-change is instantaneous. Next, we are going to calculate the explicit solution to the problem (2.1) – (2.3), for the liquid phase and, we are going to demonstrate that this solution is constant in $(0,t)$ for all t. Then we can answer affirmatively to the previous question by considering that the condition (1.10) is assumed. In order to obtain that explicit solution for the problem (2.1) – (2.3) we define the new variables and parameters

$$\begin{cases} x_* = x\sqrt{\dfrac{c_0}{k_0 t_s}} \quad , & t_* = \dfrac{t}{t_s} \\ u_*(x_*,t_*) = \dfrac{u(x,t) - T_1}{T_2 - T_1} > 0 & \\ K_{2*}(u_*) = \dfrac{K_2(u)}{k_0} \quad , & C_{2*}(u_*) = \dfrac{C_2(u)}{c_0}, \end{cases} \tag{2.4}$$

where t_s is a time scale. Following [BrTrAv], we consider the Kirchhoff transformation given by

$$\eta(x_*,t_*) = \mu\left(u_*(x_*,t_*)\right) = \int_0^{u_*(x_*,t_*)} K_{2*}(z)\,dz \tag{2.5}$$

where

$$\mu(\Psi) = \int_0^{\Psi} K_{2*}(z)\,dz \tag{2.6}$$

Next, we define the new variables

$$\begin{cases} \chi(x_*,t_*) = \displaystyle\int_0^{x_*} \frac{1}{a_2(b_2 - \eta(z,t_*))}dz \ , \quad x_* > 0 \ , \ t_* > 0 \\ \tau = t_* \\ \overline{\mu}(\chi,\tau) = \eta(x_*,t_*) \ , \chi > 0 \ , \quad \tau > 0 \, . \end{cases} \tag{2.7}$$

Now we are in condition in order to assume a similarity solution of the type

$$g(\phi) = \frac{\overline{\mu}(\chi,\tau)}{\theta} \quad , \quad \phi = \frac{\chi}{2\sqrt{\tau}} \ , \text{ with } \theta = \int_0^1 K_{2*}(z)\,dz, \tag{2.8}$$

then our problem reduce to the following conditions

$$2(\phi + \lambda)g'(\phi) + g''(\phi) = 0 \ , \ \phi > 0 \, , \tag{2.9}$$

$$g(+\infty) = 1 \ , \tag{2.10}$$

$$g'(0) = \frac{2\lambda}{\theta}\left(b_2 - \theta g(0)\right) \tag{2.11}$$

for the unknown function g, with

$$q_{0*} = \frac{q_0}{\sqrt{c_0 k_0}(T_2 - T_1)} \quad , \quad \lambda = a_2 q_{0*}. \tag{2.12}$$

The solution to the equation (2.9) and conditions (2.10)-(2.11) is given by

$$g(\phi) = A\left[\operatorname{erf}(\phi + \lambda) - \operatorname{erf}(\lambda)\right] + B \ , \quad \phi > 0 \tag{2.13}$$

where constants A and B are given by the following expressions

$$A = \frac{\lambda\,(b_2 - \theta)\,\sqrt{\pi}}{\theta\left[\exp(-\lambda^2) - \sqrt{\pi}\lambda(1 - \operatorname{erf}(\lambda))\right]} \ , \ B = 1 - \frac{\lambda\,(b_2 - \theta)\,\sqrt{\pi}(1 - \operatorname{erf}(\lambda))}{\theta\left[\exp(-\lambda^2) - \sqrt{\pi}\lambda(1 - \operatorname{erf}(\lambda))\right]} \ . \tag{2.14}$$

Then, we obtain the following result.

Theorem 1 *The parametric solution to the problem (2.1)-(2.3) is given by*

$$u(x,t) = T_1 + (T_2 - T_1)\mu^{-1}\left(\theta A\left(\operatorname{erf}\left(\frac{\chi}{2\sqrt{\tau}} + \lambda\right) - \operatorname{erf}(\lambda)\right) + \theta B\right) \ , \tag{2.15}$$

with

$$\begin{cases} x = a_2\sqrt{\dfrac{k_0 t_s \tau}{c_0}}\Big\{(b_2 - B + A\operatorname{erf}(\lambda))\,\chi - 2A\sqrt{\tau}\left[\left(\frac{\chi}{2\sqrt{\tau}} + \lambda\right)\operatorname{erf}\left(\frac{\chi}{2\sqrt{\tau}} + \lambda\right)\right. \\ \left. +\frac{1}{\sqrt{\pi}}\exp\left(-\left(\frac{\chi}{2\sqrt{\tau}} + \lambda\right)^2\right) - \lambda\operatorname{erf}(\lambda) - \frac{1}{\sqrt{\pi}}\exp\left(-\lambda^2\right)\right]\Big\} \ , \quad \chi > 0 \ , \ \tau > 0. \\ t = t_s\tau \ , \quad \tau > 0 \end{cases} \tag{2.16}$$

where A, B are defined in (2.14). Moreover, we have that

$$u(0,t) < T_1, \ \forall\, t > 0 \iff q_0 \ satisfies \ (1.11). \tag{2.17}$$

Proof.- See [BrTa2]. ■

3.- EXPLICIT SOLUTION FOR THE INSTANTANEOUS TWO-PHASE STEFAN PROCESS WITH NONLINEAR THERMAL COEFFICIENTS.

From now on we will consider the problem (1.2)-(1.8) and we will prove that it is well posed for $t > 0$ when data satisfy condition (1.11). In order to obtain the explicit solution corresponding to the problem $(1.2) - (1.8)$ we will consider the same kind of transformations used for problem $(2.1) - (2.3)$ and we define the new variables and parameters

$$\begin{cases} x_* = x\sqrt{\dfrac{c_0}{k_0 t_s}} \ , & t_* = \dfrac{t}{t_s} \\ u_{1*}(x_*, t_*) = \dfrac{u_1(x,t) - T_1}{T_2 - T_1} < 0 \ , & u_{2*}(x_*, t_*) = \dfrac{u_2(x,t) - T_1}{T_2 - T_1} > 0 \\ K_{1*}(u_{1*}) = \dfrac{K_1(u_1)}{k_0} \ , & K_{2*}(u_{2*}) = \dfrac{K_2(u_2)}{k_0} \\ C_{1*}(u_{1*}) = \dfrac{C_1(u_1)}{c_0} \ , & C_{2*}(u_{2*}) = \dfrac{C_2(u_2)}{c_0} \\ L_* = \dfrac{L}{c_0(T_2 - T_1)} \ , & y_*(t_*) = y(t)\sqrt{\dfrac{c_0}{k_0 t_s}}. \end{cases} \tag{3.1}$$

Following [BrTrAv], we consider the Kirchhoff transformation given by

$$\eta_i(x_*,t_*) = \mu_i\left(u_{i*}(x_*,t_*)\right) = \int_0^{u_{i*}(x_*,t_*)} K_{i*}(z)dz \quad , \ i=1,2 \tag{3.2}$$

where

$$\mu_i\left(\Psi\right) = \int_0^{\Psi} K_{i*}(z)dz \ , \ i=1,2\ . \tag{3.3}$$

Next, we define the new variables through the Storm transformation given by [KnPh, St]

$$\begin{cases} \chi_1(x_*,t_*) = \displaystyle\int_0^{x_*} \frac{1}{a_1(b_1-\eta_1\left(z,t_*\right))}dz \quad , \ 0<x_*<y_*(t_*) \ , \\ \chi_2(x_*,t_*) = \displaystyle\int_{y_*(t_*)}^{x_*} \frac{1}{a_2(b_2-\eta_2\left(z,t_*\right))}dz \ , \ x_*>y_*(t_*) \\ \tau = t_* \end{cases} \tag{3.4}$$

and

$$\overline{\mu_i}(\chi_i,\tau) = \eta_i(x_*,t_*) \ , \ i=1,2\ . \tag{3.5}$$

Then, the free boundary is now given by

$$S(\tau) = \chi_1(y_*(\tau),\tau) = \int_0^{y_*(\tau)} \frac{1}{a_1(b_1-\eta_1(z,\tau))}dz. \tag{3.6}$$

Owing to the condition on the free boundary and following [TrBr] we have that the interface between the two phases must move as

$$y_*(t_*) = \delta\sqrt{t_*} \ , \tag{3.7}$$

and the flux of η_2 on the free boundary takes the explicit form :

$$\frac{\partial\eta_2}{\partial x_*}\left(y_*(t_*),t_*\right) = \frac{\gamma}{\sqrt{t_*}} \ , \tag{3.8}$$

where the positive constants δ and γ must be determined. Now the free boundary $S(\tau)$ may be expressed in terms of the transformed coordinates as follows

$$S(\tau) = 2(\Lambda_1-\lambda_1)\sqrt{\tau} \ , \ \tau>0\ , \ \Lambda_1>\lambda_1>0 \tag{3.9}$$

where

$$\lambda_1 = a_1 q_{0*} \ , \qquad \Lambda_1 = a_1\gamma + \frac{\delta}{2}\left[\frac{1}{a_1b_1} + a_1L_*\right] \ . \tag{3.10}$$

Then, the problem take the following form

$$\frac{\partial\overline{\mu_1}}{\partial\tau} = \frac{\partial^2\overline{\mu_1}}{\partial\chi_1^2} + \frac{\lambda_1}{\sqrt{\tau}}\frac{\partial\overline{\mu_1}}{\partial\chi_1} \ , \ 0<\chi_1<S(\tau) \ , \ \tau>0 \ , \tag{3.11}$$

$$\frac{\partial \overline{\mu_2}}{\partial \tau} = \frac{\partial^2 \overline{\mu_2}}{\partial \chi_2^2} + \frac{\lambda_2}{\sqrt{\tau}} \frac{\partial \overline{\mu_2}}{\partial \chi_2} \quad , \quad \chi_2 > 0 \quad , \quad \tau > 0 \quad , \tag{3.12}$$

$$\overline{\mu_2}(\chi_2, 0) = \theta_2 \, , \quad \chi_2 > 0 \tag{3.13}$$

$$\overline{\mu_1}(S(\tau), \tau) = \overline{\mu_2}(0, \tau) = 0 \quad , \quad \tau > 0 \quad , \tag{3.14}$$

$$\frac{\partial \overline{\mu_1}}{\partial \chi_1}(S(\tau), \tau) \frac{1}{a_1 b_1} - \frac{\partial \overline{\mu_2}}{\partial \chi_2}(0, \tau) \frac{1}{a_2 b_2} = L_* \frac{\delta}{2\sqrt{\tau}} \, , \quad , \quad \tau > 0 \quad , \tag{3.15}$$

$$\frac{\partial \overline{\mu_1}}{\partial \chi_1}(0, \tau) = \frac{\lambda_1 \left(b_1 - \overline{\mu_1}(0, \tau) \right)}{\sqrt{\tau}} \quad , \quad \tau > 0 \quad , \tag{3.16}$$

where

$$\lambda_2 = a_2 \gamma + \frac{\delta}{2 a_2 b_2}. \tag{3.17}$$

Now we can remark that problem (3.11)-(3.16) is a two-phase Stefan problem with convective terms in both heat equations and a convective boundary condition on the fixed face. From (3.10) and (3.17) we have for the unknowns γ and δ the following relations

$$\gamma = \frac{\Lambda_1 a_1 b_1 - a_2 b_2 \lambda_2 \left[1 + a_1^2 b_1 L_* \right]}{a_1^2 b_1 \left(1 - a_2^2 b_2 L_* \right) - a_2^2 b_2} \, , \qquad \delta = \left(\lambda_2 - a_2 \gamma \right) 2 a_2 b_2. \tag{3.18}$$

If we assume a similarity solution for the problem (3.11)-(3.17) of the following type

$$\begin{cases} g_1(\phi_1) = \overline{\mu_1}(\chi_1, \tau) \quad , \quad \phi_1 = \dfrac{\chi_1}{2\sqrt{\tau}} \quad , \\ g_2(\phi_2) = \dfrac{\overline{\mu_2}(\chi_2, \tau)}{\theta_2} \quad , \quad \phi_2 = \dfrac{\chi_2}{2\sqrt{\tau}} \quad , \quad \theta_2 = \displaystyle\int_0^1 K_{2*}(z) \, dz \end{cases} \tag{3.19}$$

then it reduces to the problem (3.20)-(3.25) given by

$$2(\phi_1 + \lambda_1) g_1'(\phi_1) + g_1''(\phi_1) = 0 \quad , \quad 0 < \phi_1 < \Lambda_1 - \lambda_1 \, , \tag{3.20}$$

$$2(\phi_2 + \lambda_2) g_2'(\phi_2) + g_2''(\phi_2) = 0 \quad , \quad 0 < \phi_2 \, , \tag{3.21}$$

$$g_2(+\infty) = 1 \quad , \tag{3.22}$$

$$g_1(\Lambda_1 - \lambda_1) = g_2(0) = 0 \tag{3.23}$$

$$g_1'(0) = 2 q_{0*} a_1 \left(b_1 - g_1(0) \right) \tag{3.24}$$

$$\frac{g_1'(\Lambda_1 - \lambda_1)}{a_1 b_1} - \frac{g_2'(0) \theta_2}{a_2 b_2} = L_* \delta, \tag{3.25}$$

for the unknown functions g_1 and g_2 , and the unknown coefficients Λ_1 and λ_2.

The solution to the equations (3.20)-(3.21) and conditions (3.22)-(3.23) is given by

$$\begin{cases} g_1(\phi_1) = b_1 \dfrac{\operatorname{erf}(\phi_1 + \lambda_1) - \operatorname{erf}(\Lambda_1)}{\widetilde{g}(\lambda_1) - \operatorname{erf}(\Lambda_1)} \, , \quad 0 < \phi_1 < \Lambda_1 - \lambda_1 \\ g_2(\phi_2) = \dfrac{\operatorname{erf}(\phi_2 + \lambda_2) - \operatorname{erf}(\lambda_2)}{1 - \operatorname{erf}(\lambda_2)} \, , \quad 0 < \phi_2 \end{cases} \tag{3.26}$$

where

$$\widetilde{g}(z) = \operatorname{erf}(z) + \frac{1}{\sqrt{\pi}} \frac{\exp(-z^2)}{z} = g(z, \frac{1}{\sqrt{\pi}}) \tag{3.27}$$

and $g(z,p)$ was defined in [BrNaTa] with the following useful properties

$$\widetilde{g}(+\infty)=1\ ,\quad \widetilde{g}\,(0)=+\infty\ ,\ \widetilde{g}'\,(z)<0\ ,\ \forall\, z>0. \tag{3.28}$$

Then, the new unknowns coefficients Λ_1 and λ_2 must satisfy the following system of equations given by

$$\begin{cases} (i)\ \ \lambda_2=\dfrac{a_1b_1}{a_2b_2\left(1+a_1^2b_1L_*\right)}\left[\Lambda_1-\overline{A_1}G(\Lambda_1)\right] \\ (ii)\ \ G(\Lambda_1)=\dfrac{\theta_2}{\sqrt{\pi}a_1b_1a_2b_2}F(\lambda_2) \end{cases} \tag{3.29}$$

where

$$\overline{A_1}=a_1^2b_1\left(1-a_2^2b_2L_*\right)-a_2^2b_2 \tag{3.30}$$

and

$$\begin{cases} G(z)=\dfrac{(1+a_1^2b_1L_*)\exp(-z^2)}{\sqrt{\pi}a_1^2b_1\left[\widetilde{g}(\lambda_1)-\operatorname{erf}(z)\right]}-L_*z\ \ ,\ z>0 \\ F(z)=\dfrac{\exp(-z^2)}{1-\operatorname{erf}(z)}\ \ ,\ z>0\ . \end{cases} \tag{3.31}$$

In order to obtain the solution to our problem (3.29) we can define $\lambda_2=\lambda_2(\Lambda_1)$ from (3.29(i)).

Taking into account that L_* is from a physical point of view the inverse of the Stefan number we can obtain now the existence theorem of solution in order to have a instantantaneous phase change process for problem (1.2)-(1.8) as a function of this important physical number. Then we have the following result.

Lemma 2 *If q_0 satisfies the inequality (1.11), then the system of equations* (3.29) *admit a unique solution $\widetilde{\Lambda_1}$, $\widetilde{\lambda_2}=\lambda_2\left(\widetilde{\Lambda_1}\right)$ when data satisfies*

$$\frac{L}{c_0(T_2-T_1)}=L_*\geq\frac{1}{a_2^2b_2}\quad or\ \ 0<\frac{L}{c_0(T_2-T_1)}=L_*\leq\max\left\{0,\frac{1}{a_2^2b_2}-\frac{1}{a_1^2b_1}\right\}$$

and at least one solution $\widetilde{\Lambda_1}$, $\widetilde{\lambda_2}=\lambda_2\left(\widetilde{\Lambda_1}\right)$ when

$$\max\left\{0,\frac{1}{a_2^2b_2}-\frac{1}{a_1^2b_1}\right\}<\frac{L}{c_0(T_2-T_1)}<\frac{1}{a_2^2b_2}.$$

Proof.- It follows from the properties of functions λ_2, G and F obtained in [BrTa2]. ■

Therefore, we have obtained the following theorem in terms of the original data of the problem (1.2)-(1.8), that is:

Theorem 3 *If q_0 satisfies the inequality (1.11) then, an explicit solution to the problem (1.2)-(1.8) is given by*

$$\begin{cases} u_1(x,t)=T_1+(T_2-T_1)\mu_1^{-1}\left(b_1\dfrac{\operatorname{erf}(\frac{\chi_1}{2\sqrt{\tau}}+\lambda_1)-\operatorname{erf}(\widetilde{\Lambda_1})}{\widetilde{g}(\lambda_1)-\operatorname{erf}(\widetilde{\Lambda_1})}\right)\ , \\ \qquad 0<\chi_1<S(\tau),\ \tau>0 \\ \\ u_2(x,t)=T_1+(T_2-T_1)\mu_2^{-1}\left(\dfrac{\operatorname{erf}(\frac{\chi_2}{2\sqrt{\tau}}+\widetilde{\lambda_2})-\operatorname{erf}(\widetilde{\lambda_2})}{1-\operatorname{erf}(\widetilde{\lambda_2})}\right)\ , \\ \qquad \chi_2>0\ ,\tau>0 \end{cases}$$

with

$$\begin{cases} x = 2\dfrac{\sqrt{\dfrac{k_0 t_s \tau}{c_0}} a_1 b_1 \left(\frac{\chi_1}{2\sqrt{\tau}} + \lambda_1\right)}{\widetilde{g}(\lambda_1) - \operatorname{erf}(\widetilde{\Lambda_1})} \left[\widetilde{g}(\lambda_1) - \widetilde{g}(\frac{\chi_1}{2\sqrt{\tau}} + \lambda_1)\right] \ , \ 0 < \chi_1 < S(\tau) \ , \ \tau > 0 \\ x = \left[a_2 b_2 + \theta_2 a_2 \frac{\operatorname{erf}(\widetilde{\lambda_2})}{\operatorname{erf} c(\widetilde{\lambda_2})}\right] \chi_2 - \\ -\dfrac{2\sqrt{\tau} a_2 \theta_2}{1 - \operatorname{erf}(\widetilde{\lambda_2})} \left[\left(\frac{\chi_2}{2\sqrt{\tau}} + \widetilde{\lambda_2}\right) \widetilde{g}(\frac{\chi_2}{2\sqrt{\tau}} + \widetilde{\lambda_2}) - \widetilde{\lambda_2} \widetilde{g}(\widetilde{\lambda_2})\right] + \widetilde{\delta}\sqrt{\tau} \ , \ \chi_2 > 0 \ , \ \tau > 0 \\ t = t_s \tau \ , \qquad \tau > 0 \end{cases}$$

where $y(t) = \sqrt{\frac{k_0}{c_0}} \widetilde{\delta} \sqrt{t}$ *is the free boundary, and the coefficients* $\widetilde{\gamma}$ *and* $\widetilde{\delta}$ *are given by*

$$\widetilde{\gamma} = \frac{\widetilde{\Lambda_1} a_1 b_1 - a_2 b_2 \widetilde{\lambda_2}\left[1 + a_1^2 b_1 L_*\right]}{a_1^2 b_1 \left(1 - a_2^2 b_2 L_*\right) - a_2^2 b_2} \ , \qquad \widetilde{\delta} = \left(\widetilde{\lambda_2} - a_2 \widetilde{\gamma}\right) 2 a_2 b_2,$$

where $\widetilde{\Lambda_1}$ *and* $\widetilde{\lambda_2} = \lambda_2\left(\widetilde{\Lambda_1}\right)$ *are the solution of the system (3.29). Moreover this solution is unique when*

$$\frac{L}{c_0(T_2 - T_1)} = L_* \geq \frac{1}{a_2^2 b_2} \quad or \quad 0 < \frac{L}{c_0(T_2 - T_1)} = L_* \leq \max\left\{0, \frac{1}{a_2^2 b_2} - \frac{1}{a_1^2 b_1}\right\}.$$

ACKNOWLEDGMENTS

This paper has been partially sponsored by CONICET - UA (Rosario-Argentina). This financial support was granted to the Project PIP No. 4798/96 "Free Boundary Problems for the Heat Equation".

References

[BrTa1] Briozzo, A. C. - Tarzia, D. A, *Explicit solution of a free boundary problem for a non linear absorption model of mixed saturated-unsaturated flow*, Adv. Water Resources, 21(1998), 713-721.

[BrTa2] Briozzo, A. C. - Tarzia, D. A, *An Explicit Solution for an Instantaneous Two-Phase Stefan Problem with Nonlinear Thermal Coefficients*, Preprint (2000), Submitted to IMA J. Appl. Math.

[BrNaTa] Briozzo,A. C. - Natale, M. F. - Tarzia, D. A., *Determination of unknown thermal coefficients for Storm's type materials through a phase-change process*, Int. J. of Nonlinear Mechanics, 34 (1999), 329-340.

[BrTrAv] Broadbridge, P. - Tritscher, P. - Avagliano, A., *Free boundary problems with non-linear diffusion*, Mathl. Comput. Modeling, 18 (1993), 15-34.

[HiHa] Hill, J. M. - Hart, V. G., *The Stefan problem in nonlinear heat conduction*, J. Appl. Math. Phys. (ZAMP), 37 (1986), 206-229.

[KnPh] Knight, J. H. - Philip, J. R., *Exact solution in nonlinear diffusion*, J. Engng Math., 8 (1974), 219-227.

[NaTa] Natale, M. F. - Tarzia, D. A, *Explicit solutions to the two-phase Stefan problem for Storm's type materials*, J. Phys. A:Math. Gen., 37 (2000), 395-404.

[Ro1] Rogers, C. - Broadbridge, P., *On a class of moving boundary problems in nonlinear heat conduction: application of a Backlund transformation*, Int. J. Nonlinear Mechanics, 21 (1986), 249-256.

[Ro2] Rogers, C., *Application of a reciprocal transformation to a two-phase Stefan problem*, J. Phys. A: Math. Gen., 18 (1985), 105-109.

[SoWiAl] Solomon, A. D. - Wilson, D.G. - Alexiades, V., *Explicit solutions to phase change problems*, Quart. Appl.Math., 51 (1983), 237-243.

[St] Storm, M. L., *Heat conduction in simple metals*, J. Appl. Phys., 22 (1951), 940-951.

[Ta1] Tarzia, D. A. , *An inequality for the coefficient* σ *of the free boundary* $s(t) = 2\sigma\sqrt{t}$ *of the Neumann solution for the two-phase Stefan problem*, Quart. Appl.Math., 39 (1981/82), 491-497.

[Ta2] Tarzia, D. A., *A bibiography on moving-free boundary problems for the heat diffusion equation. The Stefan and related problems*, MAT - Serie A, # 2 (2000), (with 5869 titles on the subject, 300 pages). See www.austral.edu.ar/MAT-SerieA/2(2000)/

[TaTu] Tarzia D. A. - Turner, C. V., *A note on the existence of a waiting time for a two-phase Stefan problem*, Quart. Appl.Math; 50 (1994), 1-10.

[TrBr] Tritscher, P. - Broadbridge, P., *A similarity solution of a multiphase Stefan problem incorporating general non-linear heat conduction*, Int. J. Heat Mass Transfer; 37 (1994), 2113-2121.