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A one-phase Lamé-Clapeyron-Stefan problem with nonlinear thermal coefficients.

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Abstract

We study a one-phase Lamé-Clapeyron -Stefan problem for a semi-infinite material with nonlinear thermal coefficients with a constant temperature condition on the fixed face $x = 0$ following G. A. Tirskii, Soviet Physics Doklady, 4(1959), 288-292. We obtain sufficient conditions for data in order to have the existence of an explicit solution of a similarity type which is given by using a double fixed point.

Key words: Stefan problem, Free boundary problem, Nonlinear thermal coefficientes, Explicit solution, Nonlinear integral equation, Melting

AMS subject classification: 35R35, 80A22, 35C05, 45G10

1. Introduction.

The Lamé-Clapeyron-Stefan problem is nonlinear even in its simplest form due to the free boundary conditions [3, 4]. In particular, if the thermal coefficients of the material are temperature-dependent we have a doubly nonlinear free boundary problem. We consider the following free boundary problem (melting) for a semi-infinite material [1, 2]:

$$\rho(T)c(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right) \qquad , \qquad 0 < x < s(t) \tag{1}$$

$$T(0,t) = T_b \tag{2}$$

$$T(s(t),t) = T_m \tag{3}$$

$$k\left(T(s(t),t)\right)\frac{\partial T}{\partial x}(s(t),t) = -\rho_0\, l\, s'(t) \tag{4}$$

$$s(0) = 0 \tag{5}$$

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where $T = T(x,t)$ is the temperature of the liquid phase; $\rho(T), c(T)$ and $k(T)$ are the body's density, its specific heat, and is thermal conductivity, respectively; T_m is the phase-change temperature, $T_b > T_m$ is the temperature on the fixed face $x = 0$; $\rho_0 > 0$ is the constant density of mass at the melting temperature; $l > 0$ is the latent heat of fusion by unity of mass and $s(t)$ is the position of phase change location. This problem was firstly considered in [5] through the integral equation (13) but any mathematical result was given.

The goal of this paper is the following: we prove, in Section II, the existence of at least one explicit solution of a similarity type for the problem $(1) - (5)$ by using a double fixed point for the integral equation (13) and the trascendental equation (16) under certain hypothesis for data.

II. The one-phase Stefan problem with nonlinear thermal coefficients with constant temperature boundary condition on the fixed face.

If we define the following transformation

$$\theta(x,t) = \frac{T(x,t) - T_m}{T_b - T_m} \qquad (T(x,t) = T_m + (T_b - T_m)\theta(x,t)) \tag{6}$$

and we assume a similarity solution of the type

$$\theta(x,t) = f(\eta) \qquad , \qquad \eta = \frac{x}{2\sqrt{\alpha_0 t}} \ , \tag{7}$$

$$s(t) = 2\eta_0\sqrt{\alpha_0 t} \tag{8}$$

where η_0 is a positive parameter to be determined later, then the problem $(1) - (5)$ becomes

$$[L(f)f'(\eta)]' + 2\eta N(f)f'(\eta) = 0 \qquad , \qquad 0 < \eta < \eta_0 \tag{9}$$

$$f(0) = 1 \tag{10}$$

$$f(\eta_0) = 0 \tag{11}$$

$$f'(\eta_0) = -\frac{2\eta_0\alpha_0\rho_0 l}{k(T_m)(T_b - T_m)} \ , \tag{12}$$

where $N(T) = \dfrac{\rho(T)c(T)}{\rho_0 c_0}, L(T) = \dfrac{k(T)}{k_0}$ and k_0, ρ_0, c_0 and $\alpha_0 = \dfrac{k_0}{\rho_0 c_0}$ are the reference thermal conductivity, density of mass, specific heat and thermal diffusivity respectively. We have that the problem $(9) - (11)$ is equivalent to the following nonlinear integral equation of Volterra type:

$$f(\eta) = 1 - \frac{\Phi\left[\eta, L(f), N(f)\right]}{\Phi\left[\eta_0, L(f), N(f)\right]} \tag{13}$$

where Φ is given by

$$\Phi\left[\eta, L(f), N(f)\right] := \frac{2}{\sqrt{\pi}}\int_0^\eta \frac{1}{L(f)(t)}E(t,f)\,dt \tag{14}$$

with

$$E(x,f) := \exp\left(-2\int_0^x \frac{N(f(s))}{L(f(s))} s\, ds\right) \tag{15}$$

The condition (12) becomes

$$\frac{E(\eta_0,f)}{\Phi\left[\eta_0, L(f), N(f)\right]} = \frac{\eta_0 l\sqrt{\pi}}{c_0(T_b - T_m)} \tag{16}$$

and then the following theorem holds.

Theorem 1 *The solution of the free boundary problem* $(1)-(5)$ *is given by* (8) *and* $T(x,t) = T_m + (T_b - T_m)\, f(\eta)$, *with* $\eta = x/2\sqrt{\alpha_0 t}$ *where the function* $f = f(\eta)$ *and the coefficient* $\eta_0 > 0$ *must satisfy the nonlinear integral equation* (13) *and the condition* (16) *respectively.*■

First, in order to prove the existence of the solution of the system (13) and (16) we will obtain some preliminary results. Then we shall prove that the integral equation (13) has a unique solution for any given $\eta_0 > 0$ by using a fixed point theorem. Secondly, in order to solve the problem $(1)-(5)$ we will consider Eq. (16).

For convenience of notation, we will note $\Phi\left[\eta, f\right] \equiv \Phi\left[\eta, L(f), N(f)\right]$.

We suppose that there exists N_m, N_M, L_m, L_M positive constants such as

$$L_m \le L(T) \le L_M \quad , \quad N_m \le N(T) \le N_M \ . \tag{17}$$

We consider $C^0\left[0,\eta_0\right]$, the space of continuous real functions defined on $\left[0,\eta_0\right]$, with its norm $\|f\| = \max\limits_{\eta\in[0,\eta_0]} |f(\eta)|$.

Furthermore, we assume that the dimensionless thermal conductivity and specific heat are Lipschitz functions, i.e., there exists $\widetilde{L}$ and $\widetilde{N}$ are positive constants such that

$$|L(g) - L(h)| \le \widetilde{L}\,\|g-h\| \quad , \quad \forall g,h \in C^0\left(\mathbb{R}_0^+\right) \cap L^\infty\left(\mathbb{R}_0^+\right) \tag{18}$$

$$|N(g) - N(h)| \le \widetilde{N}\,\|g-h\| \quad , \quad \forall g,h \in C^0\left(\mathbb{R}_0^+\right) \cap L^\infty\left(\mathbb{R}_0^+\right). \tag{19}$$

Then we get:

Lemma 2 *We have*

a)

$$\exp\left(-\frac{N_M}{L_m}x^2\right) \le E(x,f) \le \exp\left(-\frac{N_m}{L_M}x^2\right) \ , \ \forall x > 0. \tag{20}$$

b) For $0 < \eta < \eta_0$

$$\tfrac{1}{L_M}\sqrt{\tfrac{L_m}{N_M}}\,\mathrm{erf}\left(\sqrt{\tfrac{N_M}{L_m}}\,\eta\right) \le \Phi\left[\eta, f\right] \le \tfrac{1}{L_m}\sqrt{\tfrac{L_M}{N_m}}\,\mathrm{erf}\left(\sqrt{\tfrac{N_m}{L_M}}\,\eta\right). \tag{21}$$

Lemma 3 *a) Let η_0 be a given positive real number. We suppose that the dimensionless thermal conductivity and specific heat verify conditions* (17), (18) *and* (19)*. Then, for all $f,\ f^* \in C^0[0,\eta_0]$ we have*

$$|E[\eta,f]-E[\eta,f^*]| \le \frac{\eta^2}{L_m}\left(\widetilde{N}+\frac{N_M\widetilde{L}}{L_m}\right)\|f^*-f\| \quad , \quad \forall \eta \in (0,\eta_0)\,. \tag{22}$$

b) Let η_0 be a given positive real number. We suppose that (17), (18) *and* (19) *holds. For all $f,\ f^* \in C^0[0,\eta_0]$, $0<\eta<\eta_0$ we have*

$$|\Phi[\eta,f]-\Phi[\eta,f^*]| \le \tfrac{2}{\sqrt{\pi}}\left(\left(\widetilde{N}+\tfrac{N_M\widetilde{L}}{L_m}\right)\tfrac{\eta^2}{3}+\widetilde{L}\right)\tfrac{\eta}{L_m^2}\|f^*-f\|\,. \tag{23}$$

Theorem 4 *Let η_0 be a given positive real number. We suppose that* (17), (18) *and* (19) *holds. If η_0 satisfies de following inequality*

$$\beta(\eta_0) := \frac{4}{\sqrt{N_m\pi}}\frac{\eta_0 L_M^{5/2} N_M \operatorname{erf}\left(\sqrt{\frac{N_m}{L_M}}\eta_0\right)}{L_m^4\operatorname{erf}^2\left(\sqrt{\frac{N_M}{L_m}}\eta_0\right)}\left(\left(\widetilde{N}+\frac{N_M\widetilde{L}}{L_m}\right)\frac{\eta_0^2}{3}+\widetilde{L}\right)<1 \tag{24}$$

then there exist a unique solution $f \in C^0[0,\eta_0]$ of the integral equation (13)*.*

Proof. Let $W : C^0[0,\eta_0] \longrightarrow C^0[0,\eta_0]$ be the operator defined by

$$W(f)_{(\eta)} = 1-\frac{\Phi[\eta,L(f)]}{\Phi[\eta_0,L(f)]} \qquad , \qquad f\in C^0[0,\eta_0]. \tag{25}$$

The solution of the equation (13) is the fixed point of the operator W, that is

$$W(f(\eta)) = f(\eta) \quad , \quad 0<\eta<\eta_0 \tag{26}$$

Let $f,f^* \in C^0[0,\eta_0]$ be, then we obtain

$$\|W(f)-W(f^*)\| = \underset{\eta\in[0,\eta_0]}{Max}|W(f(\eta))-W(f^*(\eta))|$$

$$\le \underset{\eta\in[0,\eta_0]}{Max}\left|\frac{\Phi[\eta,f^*]\Phi[\eta_0,f]-\Phi[\eta_0,f^*]\Phi[\eta,f]}{\Phi[\eta_0,f]\Phi[\eta_0,f^*]}\right|$$

$$\le A\ \underset{\eta\in[0,\eta_0]}{Max}|\Phi[\eta,f^*]\Phi[\eta_0,f]-\Phi[\eta_0,f^*]\Phi[\eta,f]| \le$$

$$\le A\ \underset{\eta\in[0,\eta_0]}{Max}\left(|\Phi[\eta,f^*]||\Phi[\eta_0,f]-\Phi[\eta_0,f^*]|+|\Phi[\eta_0,f^*]||\Phi[\eta,f^*]-\Phi[\eta,f]|\right)$$

where

$$A = \frac{N_M L_M^2}{L_m\operatorname{erf}^2\left(\sqrt{\frac{N_M}{L_m}}\eta_0\right)} > 0 \tag{27}$$

Finally, for Lemmas 2, 3 and taking into account that $0<\eta<\eta_0$, we have

$$\|W(f)-W(f^*)\| \le \beta(\eta_0)\|f^*-f\|\,.$$

Then W is a contraction operator and therefore there exists a unique solution of the integral Eq.(13) if the condition (24) is satisfied.■

Remark 1 *The solution f of the integral equation* (13), *given by the Theorem 4, depends on the real number $\eta_0 > 0$. For convenience in the notation from now on we take*

$$f(\eta) = f_{\eta_0}(\eta) = f(\eta_0, \eta) \quad , \quad 0 < \eta < \eta_0 \quad , \quad \eta_0 > 0. \blacksquare \tag{28}$$

Let Ω be the set defined by

$$\begin{aligned} \Omega &= \left\{\eta_0 \in \mathbb{R}^+ / \beta(\eta_0) < 1\right\} = \\ &= \left\{\eta_0 \in \mathbb{R}^+ / \text{there exists a solution of Eq. } (13)\right\} . \end{aligned}$$

Lemma 5 *If*

$$\frac{2L_M^2 \widetilde{L}}{L_m^3} < 1 \tag{29}$$

there exists a positive number η_0^* such that

$$\beta(\eta_0) < 1 \;\; \text{if} \;\; 0 < \eta_0 < \eta_0^* \quad , \;\; \beta(\eta_0) \geq 1 \;\; \text{if} \;\; \eta_0 \geq \eta_0^*.$$

Proof. We have $\beta(0) = \dfrac{2L_M^2 \widetilde{L}}{L_m^3}$, $\beta(+\infty) = +\infty$ and $\beta'(\eta_0) > 0 \quad \forall \eta_0 > 0$. Then $\Omega = (0, \eta_0^*)$ where $\beta(\eta_0^*) = 1.\blacksquare$

To prove the existence of the solution of the Eq.(16), we define the real function

$$H(x) := \frac{E(x, f)}{\Phi\left[x, f\right]} \;\; , \;\; x > 0 \tag{30}$$

where f is the solution of Eq.(13) given by Theorem 4.

Theorem 6 *The Eq.*(16) *has at least one solution η_0. Moreover, if x_0 is the unique solution of equation*

$$H_1(x) = \frac{xl\sqrt{\pi}}{c_0(T_b - T_m)}, \;\; x > 0, \tag{31}$$

and $x_0 < \eta_0^*$ then $\eta_0 \in \Omega$ with $\eta_0 < x_0$, where real function H_1 is defined by

$$H_1(x) := \frac{L_M \sqrt{N_M} \exp\left(\frac{-Nm}{L_M}\, x^2\right)}{\sqrt{L_m}\, \mathrm{erf}\left(\sqrt{\frac{N_M}{L_m}}\, x\right)} \;\; , \;\; x > 0 \tag{32}$$

Proof. We have

$$\text{Eq.}\,(16) \Longleftrightarrow H(x) = \frac{xl\sqrt{\pi}}{c_0(T_b - T_m)}, \;\; x > 0.$$

Therefore, there exist at least one solution $\eta_0 > 0$ of Eq.(16) because $H(x) \leq H_1(x)$ and $H(0^+) = +\infty, \;\; H(+\infty) = 0$ and then $\eta_0 < x_0$. $\blacksquare$

Remark 2 *The solution x_0 of Eq.*(31) *can be expressed as follows*

$$x_0 := M^{-1}\left(\frac{L_M \sqrt{N_M} c_0(T_b - T_m)}{l\sqrt{\pi L_m}}\right) \tag{33}$$

where

$$M(x) := x \operatorname{erf}\left(\sqrt{\frac{N_M}{L_m}}\, x\right) \exp\left(\frac{Nm}{L_M}\, x^2\right) \tag{34}$$

is an increasing real function. Then we have

$$\beta(x_0) < 1 \Longleftrightarrow \beta\left(M^{-1}\left(\frac{L_M\sqrt{N_M}c_0(T_b - T_m)}{l\sqrt{\pi L_m}}\right)\right) < 1.\blacksquare$$

And so we have the following Theorem

Theorem 7 *(i) If* N *and* L *verify only the conditions* $(17), (18), (19), (29)$ *and* $\beta(x_0) < 1$ *where* x_0 *is defined by* (33) *then there exists at least one solution of the problem* $(1) - (5)$ *where the free boundary* $s(t)$ *is given by* (8) *and the temperature is given by* $T(x,t) = T_m + (T_b - T_m)f(\eta)$, *with* $\eta = x/2\sqrt{\alpha_0 t}$ *where* f *is the unique solution of the integral equation* (13) *and* η_0 *is given by Theorem 10.*

(ii) If N *and* L *verify only the conditions* $(17),\ (18),\ (19)$ *and* (29) *then there exists at least one solution of the problem* $(1) - (5)$ *for all latent heat of fusion* $l > l_0$ *for given others parameters where* l_0 *is given by*

$$l_0 := \frac{L_M\sqrt{N_M}c_0(T_b - T_m)}{\sqrt{\pi L_m}M\left(\eta_0^*\right)}$$

where $\eta_0^* > 0$ *is characterized by the condition* $\beta\left(\eta_0^*\right) = 1.\blacksquare$

A more complete version of these results and the corresponding study for the analogous problem with a heat flux condition on the fixed face $x = 0$ instead of the temperature condition (2) will be given in a forthcoming paper.

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