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On A Two-Phase Stefan Problem with Nonlinear Thermal Coefficients *

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Abstract

We review some recent results concerning a heat conduction problem with a particular nonlinear thermal coefficients in both solid and liquid phases for a semi-infinite material $x > 0$, with phase change temperature T_1 , an initial temperature $T_2 (> T_1)$ and a heat flux of the type $q(t) = \frac{q_0}{\sqrt{t}}$ imposed on the fixed face $x = 0$.

We determine necessary and/or sufficient conditions on the parameters of the problem in order to obtain an instantaneous nonlinear two-phase Stefan problem (solidification process). We also give the corresponding explicit solution.

Resumen: Se da una revisión de algunos resultados recientes que conciernen a una ecuación del calor con particulares coeficientes térmicos no lineales en ambas fases sólida y líquida de un cuerpo semi-infinito $x > 0$, con una temperatura de cambio de fase T_1 , una temperatura inicial $T_2 (> T_1)$ y un flujo de calor del tipo $q(t) = \frac{q_0}{\sqrt{t}}$ que se impone en el borde fijo $x = 0$. Se determinan condiciones necesarias y suficientes sobre los parámetros del problema con el objetivo de obtener un problema no lineal instantáneo de Stefan a dos fases (proceso de solidificación). Se da también la correspondiente solución explícita.

Key words: Stefan problem, Instantaneous phase-change problem, Free boundary problem, Nonlinear thermal coefficients, Explicit solution.

Palabras claves: Problema de Stefan, Problema de cambio de fase instantáneo, Problemas de frontera libre, Coeficientes térmicos no lineales, Solución explícita.

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1. INTRODUCTION

We consider the two-phase Stefan problem (solidification process)[Ta2] with nonlinear thermal coefficients for a semi-infinite region $x > 0$ with phase change temperature T_1 , an

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initial temperature $T_2 > T_1$ and an imposed heat flux of the type $q(t) = \frac{q_0}{\sqrt{t}}$, ($q_0 > 0$) on the fixed face $x = 0$. For $t > 0$ we are going to determine, if there exist, the temperature distribution $u(x, t)$ and the free boundary $x = y(t)$, where

$$u(x, t) = \begin{cases} u_1(x, t) < T_1 & 0 < x < y(t) \\ T_1 & x = y(t) \\ u_2(x, t) > T_1 & x > y(t) \end{cases}, \quad (1.1)$$

which must verify the following conditions

$$C_1(u_1) \frac{\partial u_1}{\partial t} = \frac{\partial}{\partial x} \left[K_1(u_1) \frac{\partial u_1}{\partial x} \right], \quad 0 < x < y(t), \quad t > 0, \quad (1.2)$$

$$C_2(u_2) \frac{\partial u_2}{\partial t} = \frac{\partial}{\partial x} \left[K_2(u_2) \frac{\partial u_2}{\partial x} \right], \quad x > y(t), \quad t > 0, \quad (1.3)$$

$$y(0) = 0 \quad (1.4)$$

$$u_2(x, 0) = T_2 > T_1, \quad x > 0, \quad (1.5)$$

$$u_1(y(t), t) = u_2(y(t), t) = T_1, \quad t > 0, \quad (1.6)$$

$$K_1(u_1) \frac{\partial u_1}{\partial x} - K_2(u_2) \frac{\partial u_2}{\partial x} = Ly'(t), \quad \text{on } x = y(t), \quad t > 0, \quad (1.7)$$

$$K_1(u_1(0, t)) \frac{\partial u_1}{\partial x}(0, t) = \frac{q_0}{\sqrt{t}}, \quad t > 0, \quad (1.8)$$

where

x : spatial coordinate, t : time,

$u_i(x, t)$: temperature distribution for phase i ,

T_1 : phase-change or freezing temperature,

T_2 : initial temperature, L : volumetric latent heat,

$C_i(u_i)$: volumetric heat capacity for phase i ,

$K_i(u_i)$: thermal conductivity for phase i ,

$y(t)$: free boundary (solid-liquid interface) at time t ,

q_0 : positive given constant which characterizes the heat flux on the fixed face,

$i = 1$: solid phase, $i = 2$: liquid phase.

We assume that the volumetric heat capacity and the thermal conductivity for each phase i ($i = 1, 2$) are related as follow :

$$C_i(u_i) = \frac{K_i(u_i)c_0}{k_0 a_i^2 \left[b_i - \frac{1}{k_0} \int_0^{\frac{u_i - T_1}{T_2 - T_1}} K_i(T_1 + (T_2 - T_1)z) dz \right]^2} \quad (1.9)$$

with the assumption given by

$$\frac{1}{k_0(T_2 - T_1)} \int_{T_1}^{T_2} K_2(z) dz < b_2 \quad (1.10)$$

where a_i, b_i ($i = 1, 2$) are positive constants and k_0, c_0 are scales for the thermal conductivity and volumetric heat capacity respectively. The heat flux condition of the type (1.8) was firstly considered in [Ta1] where an inequality for the coefficient q_0 was found in order to have an instantaneous two-phase Stefan problem with constant thermal coefficients, for both solid and liquid phases. Other problems in this direction are given by [BrTa1, HiHa, NaTa, Ro1, Ro2, SoWiAl]. The nonlinear relations (1.9) follows from the solidification of iron on a cooper base [TrBr]. Furthermore, these relations imply that the material is of the Storm's type, that is to say [BrNaTa, HiHa, NaTa, Ro2, St]

$$\frac{1}{\sqrt{K_i(u_i)C_i(u_i)}} \frac{d}{dT} \left(\log \sqrt{\frac{C_i(u_i)}{K_i(u_i)}} \right) = \frac{a_i}{\sqrt{c_0 k_0}(T_2 - T_1)} = const. , i = 1, 2$$

The goal of this paper is to determine which conditions on the parameters of the problem (in particular q_0) must be satisfied in order to have an instantaneous phase-change process.

In Section 2 we consider the associated nonlinear heat conduction problem corresponding to the initial liquid temperature T_2 and the heat flux condition on $x = 0$ of the type $\frac{q_0}{\sqrt{t}}$ for $t > 0$. The nonlinear condition between the thermal conductivity heat capacity is supposed to be of the type (1.9). We give a necessary condition for the heat flux input coefficient q_0 , *i.e.*

$$q_0 > \frac{\sqrt{c_0 k_0}(T_2 - T_1)}{a_2} Q^{-1} \left((k_0 b_2(T_2 - T_1))^{-1} \int_{T_1}^{T_2} K_2(z) dz \right) \tag{1.11}$$

in order to obtain an instantaneous change phase process, where Q is the real function defined by

$$Q(x) = \sqrt{\pi} x \exp(x^2)(1 - \operatorname{erf}(x)) , x > 0, \tag{1.12}$$

with the properties $Q(0) = 0, Q(+\infty) = 1, Q'(x) > 0, \forall x > 0$, where the error function is given by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-w^2) dw . \tag{1.13}$$

In the Section 3 we consider the nonlinear two phase Stefan problem (1.2) – (1.8) and we prove that it admits a similarity solution if the condition (1.11) for the coefficient q_0 obtained in Section 2 is verified. Some of these results are proved in [BrTa2].

2.- A NONLINEAR HEAT CONDUCTION PROBLEM AND ITS INSTANTANEOUS PHASE-CHANGE PROCESS.

We consider a semi-infinite slab $x \geq 0$ of a material that freezes at temperature T_1 . We suppose that it is initially hot at the uniform temperature $T_2 > T_1$ and it has nonlinear heat transfer coefficients. However, what happens if a heat flux of the type $\frac{q_0}{\sqrt{t}}$ is imposed at $x = 0$? Our interest is found relations among data in order to obtain an instantaneous phase change process, that is the temperature of the material at $x = 0$ must be less than T_1 for all positive time. Then, we consider the following nonlinear heat conduction problem corresponding to the initial phase (liquid phase) given by

$$C_2(u) \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[K_2(u) \frac{\partial u}{\partial x} \right] , x > 0 , t > 0 , \tag{2.1}$$

$$u(x, 0) = T_2, \quad x > 0, \quad (2.2)$$

$$K_2(u(0, t)) \frac{\partial u}{\partial x}(0, t) = \frac{q_0}{\sqrt{t}}, \quad t > 0, \quad (2.3)$$

where K_2 and C_2 have the relation (1.9).

Then the question that follows is: Which conditions must be satisfied for the parameters q_0, T_1, T_2, K_2 and C_2 in order to have that the temperature $u(0, t) < T_1, \forall t > 0$? If the answer is affirmative we can assure [SoWiAl, Ta1, TaTu] that the phase-change is instantaneous. Next, we are going to calculate the explicit solution to the problem (2.1) – (2.3), for the liquid phase and, we are going to demonstrate that this solution is constant in $(0, t)$ for all t . Then we can answer affirmatively to the previous question by considering that the condition (1.10) is assumed. In order to obtain that explicit solution for the problem (2.1) – (2.3) we define the new variables and parameters

$$\begin{cases} x_* = x \sqrt{\frac{c_0}{k_0 t_s}}, & t_* = \frac{t}{t_s} \\ u_*(x_*, t_*) = \frac{u(x, t) - T_1}{T_2 - T_1} > 0 \\ K_{2*}(u_*) = \frac{K_2(u)}{k_0}, & C_{2*}(u_*) = \frac{C_2(u)}{c_0}, \end{cases} \quad (2.4)$$

where t_s is a time scale. Following [BrTrAv], we consider the Kirchhoff transformation given by

$$\eta(x_*, t_*) = \mu(u_*(x_*, t_*)) = \int_0^{u_*(x_*, t_*)} K_{2*}(z) dz \quad (2.5)$$

where

$$\mu(\Psi) = \int_0^{\Psi} K_{2*}(z) dz \quad (2.6)$$

Next, we define the new variables

$$\begin{cases} \chi(x_*, t_*) = \int_0^{x_*} \frac{1}{a_2(b_2 - \eta(z, t_*))} dz, & x_* > 0, \quad t_* > 0 \\ \tau = t_* \\ \bar{\mu}(\chi, \tau) = \eta(x_*, t_*) & \chi > 0, \quad \tau > 0. \end{cases} \quad (2.7)$$

Now we are in condition in order to assume a similarity solution of the type

$$g(\phi) = \frac{\bar{\mu}(\chi, \tau)}{\theta}, \quad \phi = \frac{\chi}{2\sqrt{\tau}}, \quad \text{with } \theta = \int_0^1 K_{2*}(z) dz, \quad (2.8)$$

then our problem reduce to the following conditions

$$2(\phi + \lambda)g'(\phi) + g''(\phi) = 0, \quad \phi > 0, \quad (2.9)$$

$$g(+\infty) = 1, \quad (2.10)$$

$$g'(0) = \frac{2\lambda}{\theta} (b_2 - \theta g(0)) \quad (2.11)$$

for the unknown function g , with

$$q_{0*} = \frac{q_0}{\sqrt{c_0 k_0} (T_2 - T_1)} , \quad \lambda = a_2 q_{0*}. \quad (2.12)$$

The solution to the equation (2.9) and conditions (2.10)-(2.11) is given by

$$g(\phi) = A [\operatorname{erf}(\phi + \lambda) - \operatorname{erf}(\lambda)] + B , \quad \phi > 0 \quad (2.13)$$

where constants A and B are given by the following expressions

$$A = \frac{\lambda (b_2 - \theta) \sqrt{\pi}}{\theta [\exp(-\lambda^2) - \sqrt{\pi} \lambda (1 - \operatorname{erf}(\lambda))]} , \quad B = 1 - \frac{\lambda (b_2 - \theta) \sqrt{\pi} (1 - \operatorname{erf}(\lambda))}{\theta [\exp(-\lambda^2) - \sqrt{\pi} \lambda (1 - \operatorname{erf}(\lambda))]} . \quad (2.14)$$

Then, we obtain the following result.

Theorem 1 *The parametric solution to the problem (2.1)-(2.3) is given by*

$$u(x, t) = T_1 + (T_2 - T_1) \mu^{-1} \left(\theta A \left(\operatorname{erf}\left(\frac{\chi}{2\sqrt{\tau}} + \lambda\right) - \operatorname{erf}(\lambda) \right) + \theta B \right) , \quad (2.15)$$

with

$$\begin{cases} x = a_2 \sqrt{\frac{k_0 t_s \tau}{c_0}} \left\{ (b_2 - B + A \operatorname{erf}(\lambda)) \chi - 2A \sqrt{\tau} \left[\left(\frac{\chi}{2\sqrt{\tau}} + \lambda\right) \operatorname{erf}\left(\frac{\chi}{2\sqrt{\tau}} + \lambda\right) \right. \right. \\ \left. \left. + \frac{1}{\sqrt{\pi}} \exp\left(-\left(\frac{\chi}{2\sqrt{\tau}} + \lambda\right)^2\right) - \lambda \operatorname{erf}(\lambda) - \frac{1}{\sqrt{\pi}} \exp(-\lambda^2) \right] \right\} , \quad \chi > 0 , \tau > 0 . \\ t = t_s \tau , \quad \tau > 0 \end{cases} \quad (2.16)$$

where A, B are defined in (2.14). Moreover, we have that

$$u(0, t) < T_1, \forall t > 0 \iff q_0 \text{ satisfies (1.11)}. \quad (2.17)$$

Proof.- See [BrTa2]. ■

3.- EXPLICIT SOLUTION FOR THE INSTANTANEOUS TWO-PHASE STEFAN PROCESS WITH NONLINEAR THERMAL COEFFICIENTS.

From now on we will consider the problem (1.2)-(1.8) and we will prove that it is well posed for $t > 0$ when data satisfy condition (1.11). In order to obtain the explicit solution corresponding to the problem (1.2) – (1.8) we will consider the same kind of transformations used for problem (2.1) – (2.3) and we define the new variables and parameters

$$\begin{cases} x_* = x \sqrt{\frac{c_0}{k_0 t_s}} , & t_* = \frac{t}{t_s} \\ u_{1*}(x_*, t_*) = \frac{u_1(x, t) - T_1}{T_2 - T_1} < 0 , & u_{2*}(x_*, t_*) = \frac{u_2(x, t) - T_1}{T_2 - T_1} > 0 \\ K_{1*}(u_{1*}) = \frac{K_1(u_1)}{k_0} , & K_{2*}(u_{2*}) = \frac{K_2(u_2)}{k_0} \\ C_{1*}(u_{1*}) = \frac{C_1(u_1)}{c_0} , & C_{2*}(u_{2*}) = \frac{C_2(u_2)}{c_0} \\ L_* = \frac{L}{c_0 (T_2 - T_1)} , & y_*(t_*) = y(t) \sqrt{\frac{c_0}{k_0 t_s}} . \end{cases} \quad (3.1)$$

Following [BrTrAv], we consider the Kirchhoff transformation given by

$$\eta_i(x_*, t_*) = \mu_i(u_{i*}(x_*, t_*)) = \int_0^{u_{i*}(x_*, t_*)} K_{i*}(z) dz \quad , \quad i = 1, 2 \quad (3.2)$$

where

$$\mu_i(\Psi) = \int_0^{\Psi} K_{i*}(z) dz \quad , \quad i = 1, 2 . \quad (3.3)$$

Next, we define the new variables through the Storm transformation given by [KnPh, St]

$$\left\{ \begin{array}{l} \chi_1(x_*, t_*) = \int_0^{x_*} \frac{1}{a_1(b_1 - \eta_1(z, t_*))} dz \quad , \quad 0 < x_* < y_*(t_*) \quad , \\ \chi_2(x_*, t_*) = \int_{y_*(t_*)}^{x_*} \frac{1}{a_2(b_2 - \eta_2(z, t_*))} dz \quad , \quad x_* > y_*(t_*) \\ \tau = t_* \end{array} \right. \quad (3.4)$$

and

$$\bar{\mu}_i(\chi_i, \tau) = \eta_i(x_*, t_*) \quad , \quad i = 1, 2 . \quad (3.5)$$

Then, the free boundary is now given by

$$S(\tau) = \chi_1(y_*(\tau), \tau) = \int_0^{y_*(\tau)} \frac{1}{a_1(b_1 - \eta_1(z, \tau))} dz . \quad (3.6)$$

Owing to the condition on the free boundary and following [TrBr] we have that the interface between the two phases must move as

$$y_*(t_*) = \delta \sqrt{t_*} \quad , \quad (3.7)$$

and the flux of η_2 on the free boundary takes the explicit form :

$$\frac{\partial \eta_2}{\partial x_*}(y_*(t_*), t_*) = \frac{\gamma}{\sqrt{t_*}} \quad , \quad (3.8)$$

where the positive constants δ and γ must be determined. Now the free boundary $S(\tau)$ may be expressed in terms of the transformed coordinates as follows

$$S(\tau) = 2(\Lambda_1 - \lambda_1)\sqrt{\tau} \quad , \quad \tau > 0 \quad , \quad \Lambda_1 > \lambda_1 > 0 \quad (3.9)$$

where

$$\lambda_1 = a_1 q_{0*} \quad , \quad \Lambda_1 = a_1 \gamma + \frac{\delta}{2} \left[\frac{1}{a_1 b_1} + a_1 L_* \right] . \quad (3.10)$$

Then, the problem take the following form

$$\frac{\partial \bar{\mu}_1}{\partial \tau} = \frac{\partial^2 \bar{\mu}_1}{\partial \chi_1^2} + \frac{\lambda_1}{\sqrt{\tau}} \frac{\partial \bar{\mu}_1}{\partial \chi_1} \quad , \quad 0 < \chi_1 < S(\tau) \quad , \quad \tau > 0 \quad , \quad (3.11)$$

$$\frac{\partial \bar{\mu}_2}{\partial \tau} = \frac{\partial^2 \bar{\mu}_2}{\partial \chi_2^2} + \frac{\lambda_2}{\sqrt{\tau}} \frac{\partial \bar{\mu}_2}{\partial \chi_2}, \quad \chi_2 > 0, \quad \tau > 0, \quad (3.12)$$

$$\bar{\mu}_2(\chi_2, 0) = \theta_2, \quad \chi_2 > 0 \quad (3.13)$$

$$\bar{\mu}_1(S(\tau), \tau) = \bar{\mu}_2(0, \tau) = 0, \quad \tau > 0, \quad (3.14)$$

$$\frac{\partial \bar{\mu}_1}{\partial \chi_1}(S(\tau), \tau) \frac{1}{a_1 b_1} - \frac{\partial \bar{\mu}_2}{\partial \chi_2}(0, \tau) \frac{1}{a_2 b_2} = L_* \frac{\delta}{2\sqrt{\tau}}, \quad \tau > 0, \quad (3.15)$$

$$\frac{\partial \bar{\mu}_1}{\partial \chi_1}(0, \tau) = \frac{\lambda_1 (b_1 - \bar{\mu}_1(0, \tau))}{\sqrt{\tau}}, \quad \tau > 0, \quad (3.16)$$

where

$$\lambda_2 = a_2 \gamma + \frac{\delta}{2a_2 b_2}. \quad (3.17)$$

Now we can remark that problem (3.11)-(3.16) is a two-phase Stefan problem with convective terms in both heat equations and a convective boundary condition on the fixed face. From (3.10) and (3.17) we have for the unknowns γ and δ the following relations

$$\gamma = \frac{\Lambda_1 a_1 b_1 - a_2 b_2 \lambda_2 [1 + a_1^2 b_1 L_*]}{a_1^2 b_1 (1 - a_2^2 b_2 L_*) - a_2^2 b_2}, \quad \delta = (\lambda_2 - a_2 \gamma) 2a_2 b_2. \quad (3.18)$$

If we assume a similarity solution for the problem (3.11)-(3.17) of the following type

$$\begin{cases} g_1(\phi_1) = \bar{\mu}_1(\chi_1, \tau), \quad \phi_1 = \frac{\chi_1}{2\sqrt{\tau}}, \\ g_2(\phi_2) = \frac{\bar{\mu}_2(\chi_2, \tau)}{\theta_2}, \quad \phi_2 = \frac{\chi_2}{2\sqrt{\tau}}, \quad \theta_2 = \int_0^1 K_{2*}(z) dz \end{cases} \quad (3.19)$$

then it reduces to the problem (3.20)-(3.25) given by

$$2(\phi_1 + \lambda_1)g_1'(\phi_1) + g_1''(\phi_1) = 0, \quad 0 < \phi_1 < \Lambda_1 - \lambda_1, \quad (3.20)$$

$$2(\phi_2 + \lambda_2)g_2'(\phi_2) + g_2''(\phi_2) = 0, \quad 0 < \phi_2, \quad (3.21)$$

$$g_2(+\infty) = 1, \quad (3.22)$$

$$g_1(\Lambda_1 - \lambda_1) = g_2(0) = 0 \quad (3.23)$$

$$g_1'(0) = 2q_{0*} a_1 (b_1 - g_1(0)) \quad (3.24)$$

$$\frac{g_1'(\Lambda_1 - \lambda_1)}{a_1 b_1} - \frac{g_2'(0)\theta_2}{a_2 b_2} = L_* \delta, \quad (3.25)$$

for the unknown functions g_1 and g_2 , and the unknown coefficients Λ_1 and λ_2 .

The solution to the equations (3.20)-(3.21) and conditions (3.22)-(3.23) is given by

$$\begin{cases} g_1(\phi_1) = b_1 \frac{\text{erf}(\phi_1 + \lambda_1) - \text{erf}(\Lambda_1)}{\tilde{g}(\lambda_1) - \text{erf}(\Lambda_1)}, \quad 0 < \phi_1 < \Lambda_1 - \lambda_1 \\ g_2(\phi_2) = \frac{\text{erf}(\phi_2 + \lambda_2) - \text{erf}(\lambda_2)}{1 - \text{erf}(\lambda_2)}, \quad 0 < \phi_2 \end{cases} \quad (3.26)$$

where

$$\tilde{g}(z) = \text{erf}(z) + \frac{1}{\sqrt{\pi}} \frac{\exp(-z^2)}{z} = g(z, \frac{1}{\sqrt{\pi}}) \quad (3.27)$$

and $g(z, p)$ was defined in [BrNaTa] with the following useful properties

$$\tilde{g}(+\infty) = 1 \quad , \quad \tilde{g}(0) = +\infty \quad , \quad \tilde{g}'(z) < 0 \quad , \quad \forall z > 0. \quad (3.28)$$

Then, the new unknowns coefficients Λ_1 and λ_2 must satisfy the following system of equations given by

$$\begin{cases} (i) \quad \lambda_2 = \frac{a_1 b_1}{a_2 b_2 (1 + a_1^2 b_1 L_*)} [\Lambda_1 - \overline{A}_1 G(\Lambda_1)] \\ (ii) \quad G(\Lambda_1) = \frac{\theta_2}{\sqrt{\pi} a_1 b_1 a_2 b_2} F(\lambda_2) \end{cases} \quad (3.29)$$

where

$$\overline{A}_1 = a_1^2 b_1 (1 - a_2^2 b_2 L_*) - a_2^2 b_2 \quad (3.30)$$

and

$$\begin{cases} G(z) = \frac{(1 + a_1^2 b_1 L_*) \exp(-z^2)}{\sqrt{\pi} a_1^2 b_1 [\tilde{g}(\lambda_1) - \text{erf}(z)]} - L_* z \quad , \quad z > 0 \\ F(z) = \frac{\exp(-z^2)}{1 - \text{erf}(z)} \quad , \quad z > 0. \end{cases} \quad (3.31)$$

In order to obtain the solution to our problem (3.29) we can define $\lambda_2 = \lambda_2(\Lambda_1)$ from (3.29(i)).

Taking into account that L_* is from a physical point of view the inverse of the Stefan number we can obtain now the existence theorem of solution in order to have a instantaneous phase change process for problem (1.2)-(1.8) as a function of this important physical number. Then we have the following result.

Lemma 2 *If q_0 satisfies the inequality (1.11), then the system of equations (3.29) admit a unique solution $\tilde{\Lambda}_1, \tilde{\lambda}_2 = \lambda_2(\tilde{\Lambda}_1)$ when data satisfies*

$$\frac{L}{c_0(T_2 - T_1)} = L_* \geq \frac{1}{a_2^2 b_2} \quad \text{or} \quad 0 < \frac{L}{c_0(T_2 - T_1)} = L_* \leq \max \left\{ 0, \frac{1}{a_2^2 b_2} - \frac{1}{a_1^2 b_1} \right\}$$

and at least one solution $\tilde{\Lambda}_1, \tilde{\lambda}_2 = \lambda_2(\tilde{\Lambda}_1)$ when

$$\max \left\{ 0, \frac{1}{a_2^2 b_2} - \frac{1}{a_1^2 b_1} \right\} < \frac{L}{c_0(T_2 - T_1)} < \frac{1}{a_2^2 b_2}.$$

Proof.- It follows from the properties of functions λ_2, G and F obtained in [BrTa2]. ■

Therefore, we have obtained the following theorem in terms of the original data of the problem (1.2)-(1.8), that is:

Theorem 3 *If q_0 satisfies the inequality (1.11) then, an explicit solution to the problem (1.2)-(1.8) is given by*

$$\begin{cases} u_1(x, t) = T_1 + (T_2 - T_1) \mu_1^{-1} \left(b_1 \frac{\text{erf}\left(\frac{\chi_1}{2\sqrt{\tau}} + \lambda_1\right) - \text{erf}(\tilde{\Lambda}_1)}{\tilde{g}(\lambda_1) - \text{erf}(\tilde{\Lambda}_1)} \right) , \\ \quad 0 < \chi_1 < S(\tau), \tau > 0 \\ \\ u_2(x, t) = T_1 + (T_2 - T_1) \mu_2^{-1} \left(\frac{\text{erf}\left(\frac{\chi_2}{2\sqrt{\tau}} + \tilde{\lambda}_2\right) - \text{erf}(\tilde{\lambda}_2)}{1 - \text{erf}(\tilde{\lambda}_2)} \right) , \\ \quad \chi_2 > 0, \tau > 0 \end{cases}$$

with

$$\left\{ \begin{array}{l} x = 2 \frac{\sqrt{\frac{k_0 t_s \tau}{c_0}} a_1 b_1 \left(\frac{\chi_1}{2\sqrt{\tau}} + \lambda_1 \right)}{\tilde{g}(\lambda_1) - \operatorname{erf}(\tilde{\Lambda}_1)} \left[\tilde{g}(\lambda_1) - \tilde{g}\left(\frac{\chi_1}{2\sqrt{\tau}} + \lambda_1\right) \right], \quad 0 < \chi_1 < S(\tau), \quad \tau > 0 \\ x = \left[a_2 b_2 + \theta_2 a_2 \frac{\operatorname{erf}(\tilde{\lambda}_2)}{\operatorname{erf} c(\tilde{\lambda}_2)} \right] \chi_2 - \\ - \frac{2\sqrt{\tau} a_2 \theta_2}{1 - \operatorname{erf}(\tilde{\lambda}_2)} \left[\left(\frac{\chi_2}{2\sqrt{\tau}} + \tilde{\lambda}_2 \right) \tilde{g}\left(\frac{\chi_2}{2\sqrt{\tau}} + \tilde{\lambda}_2\right) - \tilde{\lambda}_2 \tilde{g}(\tilde{\lambda}_2) \right] + \tilde{\delta} \sqrt{\tau}, \quad \chi_2 > 0, \quad \tau > 0 \\ t = t_s \tau, \quad \tau > 0 \end{array} \right.$$

where $y(t) = \sqrt{\frac{k_0}{c_0}} \tilde{\delta} \sqrt{t}$ is the free boundary, and the coefficients $\tilde{\gamma}$ and $\tilde{\delta}$ are given by

$$\tilde{\gamma} = \frac{\tilde{\Lambda}_1 a_1 b_1 - a_2 b_2 \tilde{\lambda}_2 [1 + a_1^2 b_1 L_*]}{a_1^2 b_1 (1 - a_2^2 b_2 L_*) - a_2^2 b_2}, \quad \tilde{\delta} = (\tilde{\lambda}_2 - a_2 \tilde{\gamma}) 2 a_2 b_2,$$

where $\tilde{\Lambda}_1$ and $\tilde{\lambda}_2 = \lambda_2(\tilde{\Lambda}_1)$ are the solution of the system (3.29). Moreover this solution is unique when

$$\frac{L}{c_0(T_2 - T_1)} = L_* \geq \frac{1}{a_2^2 b_2} \quad \text{or} \quad 0 < \frac{L}{c_0(T_2 - T_1)} = L_* \leq \max \left\{ 0, \frac{1}{a_2^2 b_2} - \frac{1}{a_1^2 b_1} \right\}.$$

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