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ISSN: 1515-4904

12

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Marzo 2006

UNIVERSIDAD AUSTRAL

FACULTAD DE CIENCIAS EMPRESARIALES

MAT

Serie A: CONFERENCIAS, SEMINARIOS Y TRABAJOS DE MATEMATICA

ISSN: 1515-4904
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ISSN 1515-4904

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No. 12

AN INTRODUCTION TO THE PROBLEM OF BLOW-UP FOR SEMILINEAR AND QUASILINEAR PARABOLIC EQUATIONS

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Rosario - March 2006

Abstract. In this article we present an overview of the mathematical problem of *blow-up* for semilinear parabolic equations. We describe in a simple model the main questions posed in the theory: which solutions blow up, where and how they do. We also give some introduction to the methods and tools used in the proofs. Related problems, like quasilinear equations or blow-up produced by a boundary flux, are also treated.

Resumen. En este artículo se presenta una panorámica del problema matemático de *explosión* en ecuaciones parabólicas semilineales, describiendo las principales técnicas usadas generalmente en este tipo de problemas. Con la ayuda del ejemplo clásico de la propagación del calor con reacción se estudian las principales cuestiones de la teoría, incluyendo qué soluciones explotan, dónde y cómo lo hacen. Finalmente se incluyen algunos problemas relacionados, como el caso de ecuaciones quasilineales, sistemas o el de explosión producida por reacciones frontera.

Keywords: blow-up, parabolic problems, reaction-diffusion, semilinear problems, quasilinear problems.

Parabras claves: explosión, problemas parabólicos, reacción-difusión, problemas semilineales, problemas quasilineales.

AMS Subject Classification: 35K55, 35K57, 35K65, 35B33, 35B40

These Notes are the enlarged content of a course given by Prof. Arturo de Pablo during the VII School on Energy and Mass Transfer, Free Boundary Problems, and Applications, at the Department of Mathematics of FCE-UA, Rosario, from November 28th to December 3rd, 2005. They contain the basic ideas of the mathematical problem of blow-up for semilinear parabolic equations.

The manuscript has received and accepted on March 2006

An Introduction to the Problem of Blow-up for Semilinear and Quasilinear Parabolic Equations

Arturo De Pablo

1. Introduction

One of the subareas of Applied Mathematics that has undergone a major expansion in the last century is the theoretical study of nonlinear partial differential equations of evolution type with applications to Physics, among other sciences. And a special place in the theory of such equations is occupied by the study of *unbounded solutions* or, more specifically, solutions with a singularity in a finite time. At first, these singular solutions were considered as pathological examples valid only to delimit the degree of optimality required to the data in order to get global solvability. However, for some problems of interest *all* solutions exhibit this behaviour. Even more, these solutions are related to physical phenomena, like thermal runaway or cumulation of shock waves.

Our interest in this work is to examine the possibility that solutions of certain evolution problems be regular for a time but develop suddenly a singularity. We are then in the presence of a *locally in time*, but not *globally*, well posed problem. We concentrate here on the case in which the singularity occurs because the solution ceases to be bounded in some set, and then the equation under consideration loses, in principle, its sense. This is what we mean by *blow-up*.

Blow-up phenomenon occurs in various types of nonlinear evolution equations. It occurs for Schrödinger equations, hyperbolic equations as well as parabolic equations. In this work we shall deal only with parabolic equations.

There is a rather extensive bibliography devoted to the subject of blow-up. We mention the surveys [15, 37, 48, 64], that include all the main references to works concerning the theory of blow-up of solutions to nonlinear parabolic equations, as well as to the main applications.

As to the organization of the paper, after introducing the phenomenon of blow-up, in Section 2 we describe the questions usually posed in its study, which include which solutions blow up and where and how they do. Then we pass to analyze this phenomenon in the model semilinear problem (2.8), describing heat diffusion with a source. A preliminary Section 3 is then devoted to recall the properties of the purely diffusive heat equation (without reaction) and to introduce the techniques developed in the study of blow-up. Main Sections 4 and 5 are devoted to our objective: the study of blow-up for problem (2.8). In Sections 6 and 7 we consider some generalizations by introducing other reactions or even other diffusions. In Section 8 we consider the problem of blow-up produced by a nonlinear boundary condition. Finally, in Section 9 we mention the results obtained or the conjectures made for the problem associated to a system of reaction-diffusion equations.

We think that the following pages could serve as starting point for a researcher to get introduced in the world of blow-up for semilinear and queasilinear problems, and to get an overview of the wide spectrum of problems of interest in this subject.

These notes are an extended version of the course given at Universidad Austral in Rosario, Argentina, in December 2005, in occasion of the *Energy and Mass Transfer* meeting TEM2005. We deeply thank Domingo Tarzia for all his help and encouragement.

2. Blow-up

2.1. **Elementary example. Blow-up in ODE.** Maybe the simplest example in which the phenomenon of blow-up appears is the ODE problem

$$\begin{cases} \dfrac{du}{dt} = u^2 & \text{for } t > 0, \\ u(0) = a > 0. \end{cases} \tag{2.1}$$

The unique solution is defined only in a finite interval $[0, T)$, where $T = 1/a$. It is even explicit

$$u(t) = \frac{1}{T - t},$$

and satisfies $\lim_{t \nearrow T} u(t) = \infty$. Generally speaking we consider the t-variable as *time*, and we say that the solution *blows up* in finite time $t = T$. Motivated by this example, the concept of blow-up can be described as *the phenomenon for which the solution is not globally defined because it tends to infinity in a finite time.*

A first extension consists in considering ODE of the form $\dfrac{du}{dt} = u^p$ with $p > 1$ or, more generally, in the form

$$\begin{cases} \dfrac{du}{dt} = f(u) & \text{for } t > 0, \\ u(0) = a > 0, \end{cases} \tag{2.2}$$

with $f(u) > 0$ for $u \geq a$. The blow-up condition is

$$F(z) = \int_a^z \frac{ds}{f(s)} < C \qquad \text{for every } z > a. \tag{2.3}$$

Clearly, the blow-up time is $T = F(\infty)$. This is the first non-trivial result in the theory of blow-up.

2.2. **Blow-up in PDE.** The study of blow-up is much more difficult, and interesting from the point of view of the mathematics involved, when the problem considered contains several variables, i.e., partial derivatives appear. The typical situation is a PDE in which the solution depends on some spatial variable $x \in \mathbb{R}^N$, $N \geq 1$, as well as in time, $u = u(x, t)$.

A special class of such evolution equations are the so-called *reaction-diffusion* equations, which appear in the XXth century to model processes mainly in Physics and Biology. We stress applications to Mechanics, Technology, Biophysics and Ecology. Thus we have equations in divergence form

$$\frac{\partial u}{\partial t} = \operatorname{div}\mathcal{A}(u, \nabla u, x, t) + B(u, \nabla u, x, t), \tag{2.4}$$

the prototype being the *semilinear heat equation*

$$\frac{\partial u}{\partial t} = \Delta u + f(u). \tag{2.5}$$

See for instance [9, 65]. To fix ideas we think of $u \geq 0$ as being a temperature. The term $\operatorname{div}\mathcal{A}$ in (2.4) then represents diffusion and B can model reaction or absorption as well as convection. We study the above equations in some spatial domain $\Omega \subseteq \mathbb{R}^N$. We complement our equation with an initial datum

$$u(x, 0) = u_0(x), \tag{2.6}$$

and also with some boundary condition, usually $u = 0$ at $\partial\Omega$, if Ω is not all of $\mathbb{R}^N$.

As we have said, the first step in the study must be the establishing of a local theory: the solution exists and is unique for a small time interval $0 < t < t_0$, see Theorem 3.1. If we also dispose of a procedure that guarantees that the solution exists as long as it is regular, we then obtain a maximal time of existence $0 < t < T$, $T = T_{max}$ being finite or not. The easiest way in which T can be finite is when u is bounded for every $0 < t < T$ but tends to infinity at some point(s),

$$u(\cdot, t) \in L^\infty(\Omega) \quad \forall\, 0 \le t < T, \qquad \limsup_{t \nearrow T} \|u(\cdot, t)\|_\infty = \infty. \tag{2.7}$$

Then we say that u *blows up* at T, which is the *blow-up time*. Observe that in other cases what blows up is some derivative, while u remains bounded. This is for instance the case with du/dt in (2.2) if $f(u) = 1/(1-u)$ when u reaches the value $u = 1$. This is another type of blow-up known as *quenching*, see e.g. [48]. We concentrate in this work on blow-up produced by the unbounded growth of u.

The mathematical theory of blow-up began in the sixties (of the last century), with the works of Kaplan [44], Fujita [27, 28], Friedman [25] and others. The two main models considered in those works were the semilinear heat equation (2.5) with $f(u) = u^p$ and $f(u) = e^u$, and these are also the problems in which we will focus our attention. From those days on, an increasing interest on blow-up problems has attracted a great number of researchers. The best references to begin with are the books [9] and [64]. Modern surveys are [15, 37, 50].

2.3. **Main questions.** In the study of blow-up, for instance for equation (2.5), we encounter some basic questions that must be considered. Of course the first one must be: *does it exist blow-up*? If the answer is yes the following questions are in force: *when, where and how is it produced*? In the wide literature on blow-up since the previously mentioned articles, some other questions have also revealed of interest, including what happens after blow-up and if it can be computed numerically. We present here the following list of items to be treated in the study of blow-up, a list which is now more or less standard.

(1) **Existence of blow-up**
    (a) is there blow-up for some initial datum?
    (b) which data produce blow-up?
(2) **Where does blow-up occur?**
(3) **How does blow-up occur?**
    (a) blow-up rate
    (b) profile near blow-up time
(4) **When does blow-up occur?**
(5) **Can solutions be continued after blow-up?**
(6) **In case of systems**
    (a) do all the components blow up?
    (b) do they blow up at the same points?

In this work we concentrate on showing the answers to the first three questions, as well as to present the techniques used over the years by the main authors. In particular, we study existence of blow-up to the semilinear equation (2.5) and related, and also describe the way blow-up occurs by studying where and how it appears. As to the other questions of the above list, we only make here some comments and give some references.

(4) The time at which the solutions blow-up can usually be estimated in terms of the initial data. In some cases a continuous dependence on the data can be proved [11], though examples of lack of continuity also exist, [36]. On the other hand, the striking phenomenon of instantaneous blow-up occur for the simple model of exponential reaction, cf. [58, 70].

(5) An interesting question, whose investigation started with the work [6], is if there exists some reasonable way of continuing the solution beyond blow-up. This is done by approximation with problems that do not exhibit blow-up. Thus we could have *complete* or *incomplete* blow-up, depending on wether this limit is identically infinite or not for $t > T$. The best references for the questions of continuation after blow-up are the surveys [21, 37].

(6) When dealing with systems, besides the above ones a natural additional question appears: when one component blows up, must all the other components blow up at the same time or can they stay bounded? This gives rise to *simultaneous* or *non-simultaneous* blow-up. In case of simultaneous blow-up, must they blow up at the same points? The study of non-simultaneity in blow-up systems seems to be started with [62].

Another direction in which the investigations have a great increase in the last years is in the numerical analysis of blow-up. This means to develop numerical methods that not only be able to detect the blow-up phenomenon, but also describe in an accurate way the blow-up properties mentioned. See for instance [5, 41].

In what follows we focus on questions 1–3 of the above list for the semilinear problem of heat propagation with source

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u) & \text{for } x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega. \end{cases} \tag{2.8}$$

Here $\Omega = \mathbb{R}^N$ or a bounded regular domain. In this last case we add the condition $u = 0$ on $\partial\Omega$, which is interpreted as cooling. Other boundary conditions, like zero flux heat at $\partial\Omega$, could also be imposed, but we do not consider them here. In general the results are different depending on wether Ω is bounded or not. The initial function is nonnegative and as regular as we need. Again, the results for changing sign solutions are completely different.

We assume $f(s) \geq 0$, thus we are in presence of reaction. We concentrate on the power-like reaction, $f(s) = s^p$. Some comments are also made on the exponential case $f(s) = e^s$.

3. Preliminaries

3.1. **Basic properties of the Heat Equation.** Before proceeding to the study of blow-up for the semilinear heat equation, it is convenient to review some properties of the plain diffusive heat equation, in its different settings. Thus we first consider the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u & \text{for } x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}^N. \end{cases} \tag{3.1}$$

The fundamental solution, i.e., the solution to (3.1) with a Dirac mass δ_0 at the origin as initial datum, is the Gauss kernel

$$G(x,t) = (4\pi t)^{-N/2} e^{-|x|^2/4t}. \tag{3.2}$$

We also write $G_t(x) = G(x,t)$. The properties of this function that will be of great use for us are:

- $G_t(x) > 0$ for every $x \in \mathbb{R}^N$, $t > 0$;
- $\int_{\mathbb{R}^N} G_t = 1$ for every $t > 0$;
- $\int_{\mathbb{R}^N} G_t * \varphi = \int_{\mathbb{R}^N} \varphi$ for every $t \geq 0$, $\varphi \in L^1(\mathbb{R}^N)$, where the * sign means convolution

$$f * g(x) = \int_{\mathbb{R}^N} f(y)\, g(x-y)\, dy;$$

- $\lim_{t\to 0} G_t * \varphi = \varphi$ in $L^1(\mathbb{R}^N)$, and also uniformly in $\mathbb{R}^N$ if φ is continuous and bounded.
- $G_t * G_s = G_{t+s}$ for every $t,\ s \geq 0$;
- $G_t^m = m^{-N/2}(4\pi t)^{-N(m-1)/2} G_{t/m}$ for every $m > 0$.

The solution to our problem is $u(\cdot, t) = G_t * u_0$, i.e.

$$u(x,t) = (4\pi t)^{-N/2} \int_{\mathbb{R}^N} e^{-|x-y|^2/4t} u_0(y)\, dy. \tag{3.3}$$

From this expression and the above properties of G_t we deduce for u:

- *Maximum principle. Infinite speed of propagation.* If $u_0 \geq 0$ then $u \geq 0$. In fact $u(\cdot, t) > 0$ for every $t > 0$. Even more, $u(x,t) \geq A(t) G_{t/2}(x)$.
- *Conservation of mass.* If $u_0 \in L^1(\mathbb{R}^N)$, then $\int_{\mathbb{R}^N} u_0(x)\, dx = \int_{\mathbb{R}^N} u(x,t)\, dx$ for every $t > 0$.
- *Regularizing effect.* Even if u_0 is only in $L^1(\mathbb{R}^N)$ or $L^\infty(\mathbb{R}^N)$, we have $u(\cdot, t) \in C^\infty(\mathbb{R}^N)$ for every $t > 0$.
- *Decay rate.* $\|u(\cdot, t)\|_\infty \leq c t^{-N/2} \|u_0\|_1$.

All of these properties admit clear physical interpretations, usual in a diffusion process.

Consider now the non-homogeneous problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + g, & \text{for } x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & \text{for } x \in \mathbb{R}^N. \end{cases}$$

According to Duhamel's formula, the solution to this problem is

$$u(x,t) = G_t * u_0(x) + \int_0^t G_{t-s} * g(x,s)\, ds.$$

In particular, the solution to problem (2.8) can be represented implicitly by the formula

$$u(x,t) = G_t * u_0(x) + \int_0^t G_{t-s} * f(u(x,s))\, ds. \tag{3.4}$$

Finally, some comments on the problem in a bounded domain are in order. Thus consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u & \text{for } x \in \Omega,\ t > 0, \\ u = 0 & \text{for } x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega. \end{cases} \tag{3.5}$$

Assume $u_0 \in L^2(\Omega)$. Denoting by $\{\lambda_j\}_{j\geq 1}$ the sequence of eigenvalues of $-\Delta$ in Ω, and $\{\varphi_j\}_{j\geq 1}$ the corresponding eigenfunctions, the solution to problem (3.5) can be written in the form

$$u(x,t)=\sum_{j=1}^{\infty}\widehat{u}_j\varphi_j(x)e^{-\lambda_j t},$$

where $\widehat{u}_j$ are the Fourier coefficients of u_0 in the basis $\{\varphi_j\}$. From this we see that the decay rate is different from the case of the whole space:

$$\|u(\cdot,t)\|_\infty\le e^{-\lambda_1 t}. \tag{3.6}$$

3.2. **Main tools in the study of blow-up.** We devote this section to enumerate the main tools and techniques used in the study of blow-up for problem (2.8). We begin by setting up the local theory.

Theorem 3.1. i) *Assume f is Lipschitz continuous on $\mathbb{R}_+$, $\partial\Omega\in C^{2+\alpha}$, $u_0\in C^\alpha(\overline{\Omega})$. Then there exists a unique classical solution to problem* (2.8) *defined in a maximal time interval* $[0,T_{max})$.

ii) *If $T_{max}<\infty$ then $\lim_{t\nearrow T_{max}}\|u(\cdot,t)\|_\infty=\infty$.*

Once the local existence of solution is established, our next step is to characterize when T_{max} is finite or not. This is usually done by means of comparison or multiplication by specific functions. Thus we use the following

(1) **Comparison principle**

$$\left.\begin{cases}\dfrac{\partial u}{\partial t}\le\Delta u+f(u) & \text{for } x\in\Omega,\ 0<t<T,\\ \dfrac{\partial v}{\partial t}\ge\Delta v+f(v) & \text{for } x\in\Omega,\ 0<t<T,\\ u\le v & \text{for } x\in\partial\Omega,\ 0<t<T,\\ u\le v & \text{for } x\in\Omega,\ t=0,\end{cases}\right\}\Rightarrow\ u\le v\quad \text{for } x\in\Omega,\ 0<t<T.$$

Therefore, blow-up can be established by comparing with some explicit blow-up subsolution. On the other hand, global supersolutions will lead to non blow-up.

(2) **Stationary solutions**

Of course, a main ingredient is the existence or not of stationary solutions, used in comparison,

$$\begin{cases}\Delta w+f(w)=0 & \text{in } \Omega,\\ w=0 & \text{on } \partial\Omega.\end{cases} \tag{3.7}$$

It depends on the domain and on dimension, as well as on f. For instance, in the power case $f(s)=s^p$, there always exist stationary solutions if Ω is bounded, but when $\Omega=\mathbb{R}^N$ there exist such solutions if and only if $p\ge\dfrac{N+2}{N-2}$, $N\ge3$, see [38]. When $f(s)=\mu e^s$, there exist stationary solutions if μ is small, while there do not exist stationary solutions if μ is large. This is an easy exercise.

(3) **Test functions**

Functions satisfying $\Delta\varphi+c\varphi\ge0$ for some $c>0$ are revealed to be of great use. In particular, when Ω is bounded we consider the first eigenfunction of $-\Delta$ in Ω, a function $\varphi_1>0$ such that

$$\begin{cases}\Delta\varphi_1+\lambda_1\varphi_1=0 & \text{in } \Omega,\\ \varphi_1=0 & \text{on } \partial\Omega,\end{cases} \tag{3.8}$$

with $\lambda_1 = \lambda_1(\Omega) > 0$, $\int_\Omega \varphi_1 = 1$. When $\Omega = \mathbb{R}^N$ we use a gaussian, see (3.2),

$$G_k(x) = (4\pi k)^{-N/2} e^{-|x|^2/4k}, \tag{3.9}$$

which satisfies $\int_{\mathbb{R}^N} G_k = 1$, and also

$$\Delta G_k + \frac{N}{2k} G_k \geq 0. \tag{3.10}$$

(4) **Strong comparison**

When dealing with blow-up solutions, an important property is strong comparison:

let u and v be two solutions that blow up at times T_u and T_v respectively. Then

$$u \geq v,\ u \not\equiv v \quad \Rightarrow \quad T_u \leq T_v - \epsilon.$$

In other words, if $T_u = T_v$ then u and v must intersect.

(5) **Intersection comparison**

Strong comparison can be complemented with comparison of intersections, a technique that, though only valid in one dimension, has been widely applied even in quasilinear problems. It is based on counting the number of intersections between two solutions, i.e, the number of sign changes of the difference. The *Sturmian argument* says that this number cannot increase in time in the interior of the domain if it does not increase on the border. By the previous strong comparison principle, for two solutions having the same blow-up time this number cannot be zero. Thus, if they intersect once, this intersection must remain in time. See e.g. [64] or [30].

4. Semilinear problem. Critical exponents

4.1. **Existence of blow-up solutions.** We are now in a position to attack our reaction-diffusion problem (2.8). A necessary condition to have blow-up is easy.

Theorem 4.1. *We have*

$$\int^\infty \frac{ds}{f(s)} = \infty \quad \Rightarrow \quad u \textit{ is bounded for each } t < \infty.$$

Proof: Compare with the supersolution $U(t)$ given implicitly by

$$\int_a^{U(t)} \frac{ds}{f(s)} = t, \qquad a = \|u_0\|_\infty.$$

□

For instance if $f(u) = u^p$ we get no blow-up when $p \leq 1$.

The first positive result on blow-up is due to Kaplan in 1963 for problem (2.8), see [44]. Kaplan's argument, adapted later to a wide variety of equations, consists in multiplying the equation by certain convenient function and integrate over the domain, getting in this way an ODE to which condition (2.3), or similar, can be applied.

Theorem 4.2. *Assume f convex and satisfying $\int^\infty \frac{ds}{f(s)} < \infty$. Then if u_0 is large enough the solution to problem* (2.8) *blows up in finite time.*

The meaning of "u_0 large" will become clear after the proof. On the other hand, and depending on the reaction function, this condition cannot be avoided, as we will see.

PROOF: Assume Ω bounded. Multiplying the equation by the first eigenfunction φ_1 of (3.8), and defining the function $J(t) = \int_\Omega u\,\varphi_1$, we get that this function satisfies

$$\begin{aligned} J' &= \int_\Omega \Delta u\,\varphi_1 + \int_\Omega f(u)\,\varphi_1 \geq -\lambda_1 \int_\Omega u\,\varphi_1 + f\Big(\int_\Omega u\,\varphi_1\Big) \\ &= -\lambda_1 J + f(J) \equiv H(J). \end{aligned}$$

We use integration by parts and Jensen's inequality. If now $J(0) = \int_\Omega u_0\,\varphi_1$ is large, bigger than the biggest root J_0 of H, then $J' \geq c\,f(J)$ since f is superlinear. This implies that J blows up, and therefore u. Observe that $0 \leq J(t) \leq \|u(\cdot,t)\|_\infty$. If Ω is not bounded we take a bounded ball inside Ω to which apply the above argument. □

In the case $f(u) = u^p$, $p > 1$, we need $\int_\Omega u_0\,\varphi_1 > \lambda_1^{1/(p-1)}$. Thus we have that there exist blow-up solutions (when u_0 is large) if and only if $p > 1$. We call $p_0 = 1$ the *global existence exponent*.

When $f(u) = \mu e^u$, we have that $H(J) = -\lambda_1 J + \mu e^J$, which gives $H(J) > 0$ for every $J \geq 0$ whenever $\mu > \lambda_1 e$. Thus in this case there exists blow-up if u_0 is large, as before, but also for every u_0 if Ω or μ are large. Clearly, if $\Omega = \mathbb{R}^N$ all solutions blow up.

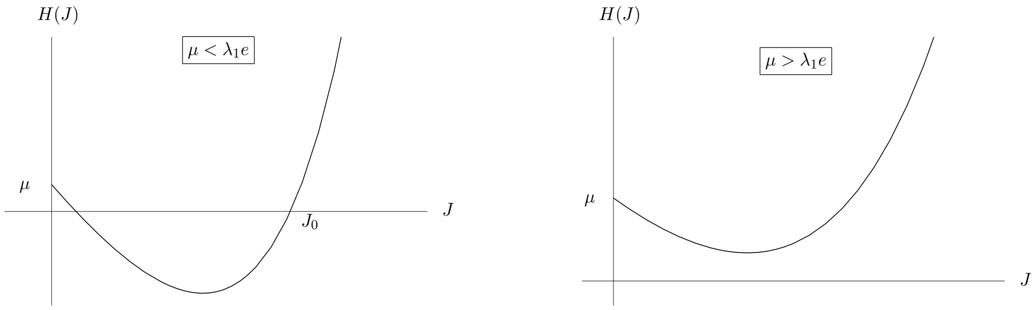

FIGURE 1. Exponential case. Blow-up for large initial data if μ (or Ω) is small or blow-up for every initial data if μ (or Ω) is large

This means that blow-up is more likely to occur in large domains, where the cooling effect of the boundary condition is smaller. But also the geometry is crucial. For instance, in the exponential case it has been proved that for the initial value $u_0 \equiv 0$, among all domains of given measure the blow-up time is smallest for the ball, [4].

In the opposite direction, if the initial datum is small, in the power case or in the exponential case provided μ is small, then the solution does not blow up. This follows from the existence of stationary solutions. For general reactions we have

Proposition 4.1. *If Ω is bounded and $f(s) \leq \lambda_1 s$ for s small, then there exist initial values for which the solution to problem* (2.8) *does not blow up.*

PROOF: If $\varepsilon > 0$ is small, the function $\psi = \varepsilon\varphi_1$ is a stationary supersolution, since

$$\Delta\psi + f(\psi) = -\lambda_1\psi + f(\psi) \leq 0 = \frac{\partial\psi}{\partial t}. \qquad \square$$

In summary, if $f(u) = u^p,\ p > 1$, and Ω is bounded, then the solution blows up provided the initial datum u_0 is large; it is globally defined if u_0 is small. This is the combined effect of diffusion and the cooling of the boundary. *What happens if $\Omega = \mathbb{R}^N$?* We know that stationary solutions exist only for large p and dimensions $N \geq 3$. There always exist global solutions if the initial value is small? We have just seen that this is not the case with the exponential reaction.

We thus pass to study the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + u^p & \text{for } x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}^N, \end{cases} \tag{4.1}$$

and focus on the existence or not of global solutions depending on p, for $p > 1$.

4.2. **Fujita exponent.** The next cornerstone in the blow-up theory was the fundamental work of Fujita in 1966, see [27].

Theorem 4.3. *If* $1 < p \le 1 + 2/N$ *every solution to problem* (4.1) *blows up.*

PROOF: If in the Kaplan's argument we use a gaussian G_k as test function, we have, for the function $J = \int_{\mathbb{R}^N} u\, G_k$, the inequality

$$J' \ge -\frac{N}{2k} J + J^p.$$

Thus the solution blows up provided $\int_{\mathbb{R}^N} u_0\, G_k \ge (N/2k)^{1/(p-1)}$. This means

$$\int_{\mathbb{R}^N} u_0\, e^{-|x|^2/4k} \ge c k^{N/2-1/(p-1)}.$$

This holds for every u_0, with k large, if we assume $1/(p-1) - N/2 > 0$. This implies blow-up for every solution in the subcritical case $p < 1 + 2/N$.

In the critical case $p = 1 + 2/N$, the above gives blow-up whenever $\int_{\mathbb{R}^N} u_0\, e^{-|x|^2/4k}$ is large. Since this integral approaches $\int_{\mathbb{R}^N} u_0$ when k is large, it suffices to have $\int_{\mathbb{R}^N} u_0$ large. On the other hand, since we can take $u(\cdot, t)$ for any $t > 0$ as new initial datum, we are thus reduced to prove that $\int_{\mathbb{R}^N} u(x,t)\, dx$ is large for t large.

Suppose u_0 is above some gaussian, $u_0 \ge AG_\alpha$, A, $\alpha > 0$. This holds if we let pass some time, as we have seen in the previous section. Then, from (3.4),

$$u(x,t) \ge G_t * u_0(x) \ge AG_t * G_\alpha(x) = AG_{t+\alpha}(x).$$

Now, again from (3.4) and the properties of the Gauss kernel,

$$\begin{aligned} \int_{\mathbb{R}^N} u(x,t)\, dx &\ge \int_0^t \int_{\mathbb{R}^N} G_{t-s} * (AG_{s+\alpha})^p(x)\, dx\, ds \\ &= A^p \int_0^t \int_{\mathbb{R}^N} G^p_{s+\alpha}(x)\, dx\, ds \\ &= A^p \int_0^t \int_{\mathbb{R}^N} (4\pi(s+\alpha))^{-N(p-1)/2} p^{-N/2} G_{(s+\alpha)/p}(x)\, dx\, ds \\ &= c \int_0^t (s+\alpha)^{-1}\, ds. \end{aligned}$$

We end by observing that this integral diverges for $t \to \infty$. □

The number $p_c = 1 + 2/N$ is called the *Fujita exponent.* We remark that the critical case $p = p_c$ in the above theorem was left open by Fujita. It was proved in the next decade by Hayakawa [42] ($N = 1, 2$) and later by Kobayashi, Sirao and Tanaka [45] ($N \geq 1$). The above easy proof is due to Weissler [74].

Fujita exponent can be understood in different ways. In [34] the authors propose a simple method of obtaining this exponent and then apply this method to different reaction-diffusion problems, see also [15]. If we equal the decay rate given by diffusion to the blow-up rate driven by reaction we get p_c. If $p > 1$ the solution blows up if it accumulates enough mass (Kaplans' argument), and this is the case of all solutions if $1 < p \leq p_c$. On the contrary, if $p > p_c$ diffusion does not allow small initial values to grow, and in fact the solutions tend to zero.

The solutions to the purely diffusive heat problem (3.1) decay as t^{-a_d}, with $a_d = N/2$; the blow-up rate is obtained just by solving the ODE $u' = u^p$, thus getting $a_r = 1/(p-1)$. We obtain Fujita exponent by imposing $a_d = a_r$. See also next section where Fujita exponent is obtained for a quasilinear diffusion equation.

The fact that when $p > p_c$ there exist small global solutions is easily proved by comparison with a supersolution. For instance we take

$$w(x,t) = \varepsilon(t+1)^{\eta} G_{t+1}(x),$$

with $\eta > 0$ small satisfying $(p-1)\eta < (p-1)N/2 - 1$. Then solutions with initial datum satisfying $u_0 \leq \varepsilon G_1$ are globally defined. Observe that the condition $p > p_c$ is crucial in the argument. We have thus proved

Theorem 4.4. *If $p > 1 + 2/N$ then the solution to problem* (4.1) *can be global or can blow up depending on the initial datum.*

Another way of proving Fujita's result is the *energy method* introduced by Levine in [48, 52]. The advantage of this method lies in the fact that it also gives a quantitative criterion of blow-up in terms of the initial data when not all the solutions blow up, i.e., in the case of Theorem 4.4.

Define the following quantity, sometimes called *energy*

$$E_u(t) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(x,t)|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} u^{p+1}(x,t)\,dx. \tag{4.2}$$

It is easy to check that $E_u(t)$ es nonincreasing when u is a solution to problem (4.1). The characterization of blow-up in terms of the energy is given below. The proof of this result is very technical, see [48]. We show instead its application in proving Fujita's result.

Theorem 4.5. *If there exists $t_0 \geq 0$ such that $E_u(t_0) < 0$ then u blows up in a finite time $T > 0$. Moreover $\lim_{t \nearrow T} E_u(t) = -\infty$.*

Another proof of Theorem 4.3 in the subcritical case: Observe that the gaussians G_k have negative energy if $k > 0$ is large enough

$$\begin{aligned} E_{G_k} &= c_1 \int_0^\infty k^{-(N+2)} z^2 e^{-z^2/2k}\,dz - c_2 \int_0^\infty k^{-N(p+1)/2} e^{-(p+1)z^2/4k}\,dz \\ &= c_1 k^{-(N+1/2)} - c_2 k^{-(N(p+1)-1)/2}, \end{aligned}$$

provided $N + 1/2 > (N(p+1)-1)/2$, i.e., $p < 1 + 2/N$.

Therefore, applying Theorem 4.5, the solution with initial value G_k must blow up, and also any initial datum above G_k must produce blow-up. But k is large, so we can always have $u_0 \geq G_k$, and then every solution blows up in this range. □

If $p = 1 + 2/N$ we deduce the result by looking more carefully at the coefficients. In bounded domains the condition of negative energy is also sufficient.

5. Semilinear problem. Asymptotic behaviour

5.1. **Blow-up rates.** In this section we begin the description of the blow-up solutions close to the blow-up time by studying the speed at which these solutions blow up. We study here the blow-up rates, the blow-up sets and the blow-up profiles, for a solution u to problem (2.8) that blows up at a certain time $T > 0$. Again we concentrate on the case $f(u) = u^p$, and end with some comments on the case $f(u) = e^u$.

At first sight, since blow-up is produced by the reaction term, diffusion must be small near blow-up points and blow-up time. This would mean that at those points the solution must be close to satisfy the ODE $w' = f(w)$. The solution to this equation (assuming w blows up at time T) is given by the formula

$$F(w(t)) \equiv \int_{w(t)}^{\infty} \frac{ds}{f(s)} = T - t. \tag{5.1}$$

Thus the expected result for our blow-up solution u would be

$$\|u(\cdot,t)\|_\infty \sim G(T-t) \qquad \text{for } t \sim T, \tag{5.2}$$

where G is the inverse function of F. An estimate of this kind is called *blow-up rate.* When $f(u) = u^p$, $p > 1$, we have $G(s) = cs^{-1/(p-1)}$, i.e.,

$$\|u(\cdot,t)\|_\infty \sim (T-t)^{-1/(p-1)} \qquad \text{for } t \sim T. \tag{5.3}$$

When the above holds we say that the rate is the *natural* or *self-similar* one, or that blow-up is of *type I.* We show here that the rate is in general the natural one, with some restrictions on the exponents, [26, 40]. Without these restrictions blow-up can be of *type II* (also called *superfast blow-up*), i.e.

$$\limsup_{t\to T}(T-t)^{1/(p-1)}\|u(\cdot,t)\|_\infty = \infty. \tag{5.4}$$

We begin by observing that a lower bound of the rate is easy to obtain. For simplicity assume u_0 symmetric with maximum at the origin. Then $M(t) = \|u(\cdot,t)\|_\infty = u(0,t)$. Clearly $\Delta u(0,t) \le 0$. Then

$$M'(t) = \frac{\partial u}{\partial t}(0,t) \le f(u(0,t)) = f(M(t)).$$

Integration of this inequality gives $F(M(t)) \ge T - t$. Another proof, when $\Omega = \mathbb{R}^N$, uses strong comparison of the solution u with the planar solution w given in (5.1). Having both the same blow-up time they must intersect, and thus it holds $M(t) > w(t)$.

The first result in the direction of proving the upper estimate in (5.3) was established by Weissler in [74] under the assumption $p < p_S$, where $p_S = (N+2)/(N-2)$ if $N \ge 3$, $p_S = \infty$ if $N = 1, 2$ is the Sobolev critical exponent. He also needed to impose Ω a ball, u radially decreasing in space and nonincreasing in time. Without the restriction of radiallity the estimate was obtained by Friedman and McLeod in [26]. Giga and Kohn eliminated the hypothesis $\dfrac{\partial u}{\partial t} \ge 0$, see [40]. They also considered the case $\Omega = \mathbb{R}^N$. In the critical case $p = p_S$ the natural rate is proved in [23] for radially symmetric and nonincreasing solutions. See also [54].

The proof in [40] of the upper bound of the rate is based on a very powerful scaling technique, using similarity variables. In fact, these authors introduced this technique in order to establish the asymptotic behaviour, as we will see in the next section.

Theorem 5.1. *Let Ω be a bounded convex domain in $\mathbb{R}^N$, or else $\Omega = \mathbb{R}^N$. Then*

$$\|u(\cdot,t)\|_\infty \leq c(T-t)^{-1/(p-1)} \tag{5.5}$$

provided $1 < p < (N+2)/(N-2)$ *or* $N \leq 2$.

Consider the new function h defined by

$$h(\xi,\tau) = (T-t)^\alpha u(x,t), \qquad \xi = x(T-t)^{-1/2},\ \tau = -\log(T-t), \tag{5.6}$$

where $\alpha = 1/(p-1)$. This is called a self-similar change of variables. The upper bound (5.5) is equivalent to $h \leq c$. This is why α is called the self-similar rate. Moreover, h is defined for every τ, since $t = T$ corresponds to $\tau = \infty$. The function h solves the equation

$$\frac{\partial h}{\partial \tau} + \alpha h + \frac{1}{2}\,\xi \cdot \nabla h = \Delta h + h^p. \tag{5.7}$$

To understand why h must be bounded, suppose that it were to become large for some τ. Then the reaction term h^p would begin to dominate, making h blow up in finite time. But this is a contradiction. It follows (heuristically) that h is never too large, see for instance [56]. This argument fails if $p \geq p_S$, since there exist unbounded global solutions to equation (5.7).

An entirely different argument used also in [40] is the following: suppose that u were to blow up faster (for instance at the origin) than the self-similar rate, say $u(0,t) \sim (T-t)^{-\gamma}$, for some $\gamma > \alpha$. Then the right rescaled function to study would be not h but

$$g(\eta,\sigma) = (T-t)^\gamma u(x,t), \qquad \eta = x(T-t)^{-\gamma/2\alpha},\ \sigma = \frac{1}{\gamma/\alpha - 1}(T-t)^{1-\gamma/\alpha}.$$

It satisfies the equation

$$\frac{\partial g}{\partial \sigma} + \frac{1}{(\gamma/\alpha - 1)\sigma}\Big(\gamma g + \frac{\gamma}{2\alpha}\,\eta \cdot \nabla g\Big) = \Delta g + g^p.$$

It also satisfies $g(0,\sigma) \sim 1$. Letting $\sigma \to \infty$, we expect g to be asymptotically stationary. Since the second term in the left-hand side also tends to zero, we get in the limit a nonnegative global solution $\widetilde{g}$ of

$$\Delta \widetilde{g} + \widetilde{g}^p = 0 \qquad \text{in } \mathbb{R}^N,$$

with $\widetilde{g}(0) > 0$. This is a contradiction since no such solution exists in the range of parameters considered, as we have said in the previous section, see [38]. The detailed proof of this argument uses the rescaling

$$g(x,t) = \frac{1}{\mu} u(\mu^{-\alpha/2}x, \mu^{-\alpha}t + s), \qquad x \in \Omega,\ t \in (-s\mu^2, 0),$$

where $0 < s < T$ is fixed and $\mu = u(0,s) = \|u(\cdot,s)\|_\infty$. With the assumption that (5.5) does not hold it is proved the convergence as $\mu \to \infty$ to a stationary solution. The above mentioned contradiction finishes the proof.

The first counterexample of the natural rate for supercritical exponents $p \geq p_S$, i.e., the first example of superfast blow-up, is due to Herrero and Velázquez [43] (in fact for dimensions $N \geq 11$ and very large p). Later an example of superfast blow-up is obtained for $p = p_S$, but with changing sign solutions, [23]. We remark that the study of superfast blow-up, or blow-up of type II, is nowadays a very active branch in blow-up theory. See the recent papers [54, 55].

In the exponential case $f(u) = e^u$, it is proved in [26] the estimate

$$C_1 - \log(T-t) \le \|u(\cdot,t)\|_\infty \le C_2 - \log(T-t) \qquad \text{for } t \sim T, \tag{5.8}$$

for Ω bounded, and under the assumption $\dfrac{\partial u}{\partial t} \ge 0$, where $-\infty < C_1 < C_2 < \infty$.

5.2. **Asymptotic profile.** We now study the asymptotic behaviour by describing the final shape of the solution, the so-called *asymptotic profile*. This is done by introducing the similarity variables (5.6) of Giga and Kohn [40]. Our purpose is to prove the convergence of the rescaled solution h to a stationary state. We thus consider equation (5.7), that we write down here again for convenience.

$$\frac{\partial h}{\partial \tau} + \alpha h + \frac{1}{2}\xi \cdot \nabla h = \Delta h + h^p. \tag{5.9}$$

Recall that $\alpha = 1/(p-1)$. The first result is proved in [39].

Theorem 5.2. *If $p \le p_S$ the unique nontrivial nonnegative bounded stationary solution to equation* (5.9) *is the constant $H = \alpha^\alpha$.*

Observe that this constant comes from the equality $\alpha h = h^p$, $h > 0$. In the case $p > p_S$ a family of nonconstant stationary solution exist. Dropping the condition of being bounded, a singular stationary solution exists if $p > N/(N-2)$. We remark by passing that the existence of these special solutions is the starting point in the study of superfast blow-up.

Thus, proving a stabilization result to H means, in the original variables

Theorem 5.3. *Under the above hypotheses it holds*

$$\lim_{t\to T}(T-t)^\alpha u(x,t) = \lim_{\tau\to\infty} h(xe^{\tau/2},\tau) = H, \tag{5.10}$$

uniformly in sets of the form $\{|x| \le c(T-t)^{1/2}\}$.

The ingredients required to obtain this result are:

- compactness, so we can take subsequential limit;
- a Lyapunov function to show that $\dfrac{\partial h}{\partial \tau}(\xi, \tau+\tau_j)$ converges to zero in some weak sense;
- since the nontrivial positive stationary solution is unique then the convergence is true for every subsequence $\tau_j \to \infty$.

Proof of Theorem 5.3: Consider the Lyapunov functional (energy in rescaled variables)

$$\mathcal{L}_h(\tau) = \int_{\mathbb{R}^N} \Big(\frac{1}{2}|\nabla h|^2 - \frac{1}{p+1}h^{p+1} + \frac{\alpha}{2}h^2\Big)\rho,$$

with the weight $\rho(\xi) = e^{-|\xi|^2/4}$. From the equation we see that this energy is nonincreasing along the orbits,

$$\frac{d}{d\tau}\mathcal{L}_h(\tau) = -\int_{\mathbb{R}^N} \Big(\frac{\partial h}{\partial \tau}\Big)^2 \rho \le 0.$$

This implies, using the rates,

$$0 < c_1 \le \mathcal{L}_h(\tau) \le c_2\,.$$

Also, integrating,

$$\int_{\tau_1}^{\tau_2}\int_{\mathbb{R}^N}\Big(\frac{\partial h}{\partial \tau}\Big)^2\rho = \mathcal{L}_h(\tau_1)-\mathcal{L}_h(\tau_2)\le c.$$

Define, for a sequence $\tau_j\to\infty$, the function $h_j(\xi,\tau)=h(\xi,\tau+\tau_j)$. Let $\widetilde{h}=\lim_{j\to\infty}h_j$.

The above estimates imply

$$\int_0^1\int_{\mathbb{R}^N}\Big(\frac{\partial h_j}{\partial \tau}\Big)^2\rho\le\int_{\tau_j}^{\infty}\int_{\mathbb{R}^N}\Big(\frac{\partial h}{\partial \tau}\Big)^2\rho\to 0,$$

and so $\widetilde{h}$ does not depend on τ. Passing to the limit in the equation, we obtain that $\widetilde{h}$ is a stationary solution. It could be $\widetilde{h}=0$, but the blow-up rates imply $\widetilde{h}\ge c>0$. Thus $\widetilde{h}=H$. □

If the set of stationary solutions were not unique, nor a discrete set, different subsequential limits might converge to different stationary solutions. The convergence result must then be written in terms of ω-limits. This occurs in quasilinear problems.

A more accurate description of the asymptotic behaviour is performed by Velázquez in [73], by obtaining the second term in the approximation. He uses linearization of equation (5.9) around H, and thus Hermite polynomials appear in this description. See the mentioned paper for the precise formulas.

We again comment by passing the analogous result to Theorem 5.3 corresponding to the exponential case. We have here

$$\lim_{t\to T}(T-t)e^{u(x,t)}=1 \tag{5.11}$$

uniformly in sets $\{|x|\le c(T-t)^{1/2}\}$, see for instance [53].

5.3. **Blow-up sets.** The description of the set of points where the solution blows up, the *blow-up set*

$$B(u)=\{x\in\Omega\,:\,\exists\, x_n\to x,\ t_n\to T^-\text{ with } u(x_n,t_n)\to\infty\}, \tag{5.12}$$

is in general a delicate matter, even more in dimensions $N>1$, see [72]. The first results, see [26], are the following. We fix the power reaction $f(u)=u^p$, $p>1$.

Theorem 5.4. i) *If Ω is convex and $\dfrac{\partial u}{\partial t}\ge 0$ then $B(u)$ is a compact subset of Ω.*

ii) *If Ω is a ball and u_0 is radially decreasing, then $B(u)$ reduces to the origin.*

ii) *If $N=1$ and u_0' changes sign only once, then $B(u)$ reduces to a point.*

The first statement asserts that blow-up takes place away from the boundary. This seems clear since we fix value zero at $\partial\Omega$. The proof of the easiest radial case is based on the estimate

$$u(x,t)\le c|x|^{-2/(p-1)}, \tag{5.13}$$

which gives $u(x,t)$ bounded for every $0<t<T$ at every fixed point $x\ne 0$. To obtain this estimate consider the auxiliary function

$$J(r,t)=r^{N-1}\frac{\partial u}{\partial r}+\varepsilon r^{N+\delta}u^{\gamma}(r,t) \tag{5.14}$$

where $r = |x| \in [0, R]$, ε, $\delta > 0$ are small and $1 < \gamma < p$. Via maximum principle (prove that $\dfrac{\partial J}{\partial t} - \Delta J - kJ \le 0$), it can be shown that $J \le 0$ for every $r \in [0, R]$, $0 \le t < T$, i.e.

$$u^{-\gamma}\frac{\partial u}{\partial r} \le -\varepsilon r^{1+\delta}, \tag{5.15}$$

Integration of this inequality gives $u(r,t) \le cr^{-(2+\delta)/(\gamma-1)}$. Passing to the limit $\delta \to 0$, $\gamma \to p$ gives the desired conclusion.

More generally, and first in dimension $N = 1$, if u_0 has a finite number m of maxima, then the blow-up set is a discrete set containing at most m elements, see [12]. For $N > 1$, in [72] it is proved that, assuming the solution satisfies the natural rate, the $(N-1)$-dimensional Hausdorff measure of $B(u)$ is bounded in compact sets of $\mathbb{R}^N$.

The fact that the blow-up set in the radial case for the exponential reaction is a unique point has been proved also in [26], by showing that u satisfies

$$u(x,t) \le C_1 - C_2 \log |x|. \tag{5.16}$$

6. Other types of reaction

In extending the results of the preceding two sections to the more general problem (2.4), a great variety of different reaction terms B have been considered in the literature of blow-up. We present here some examples. More general diffusion terms are considered in next section.

Besides $f(u) = u^p$ and $f(u) = e^u$, it is interesting to consider the function $f(u) = (1+u)\log^r(1+u)$, $r > 1$, in problem (2.8). First of all, it is (slightly) superlinear for large values of u. Therefore there exist blow-up solutions, Theorem 4.2. On the other hand, $f(u) \sim u^r$ when u is small; thus all solutions blow up if $1 < r \le 1 + 2/N$, Theorem 4.3. But what makes different this case is the asymptotic behaviour of blow-up solutions. As to the blow-up set, it is proved in [35] (see also [46]), that if $1 < r < 2$, then all points of $\mathbb{R}^N$ are blow-up points. Also, if $r = 2$ then $B(u) \supseteq \{|x| \le \pi\}$; in the one dimensional case it is exactly $B(u) = \{|x| \le \pi\}$. If $r > 2$ blow-up reduces to the origin. This striking property when $1 < r \le 2$ is common in problems with quasilinear diffusion operators, as we will see in next section, but not with linear diffusion. The exponent $r = 2$ is called the *localization exponent*. More surprising is the asymptotic profile. The complete equation does not admit any exact self-similar or invariant solution. In the above paper [35] it is proved that for t close to the blow-up time, an effect of asymptotic simplification takes place: the solution resembles the solution to another equation, namely the *Hamilton-Jacobi* type equation

$$\frac{\partial U}{\partial t} = \frac{|\nabla U|^2}{1+U} + (1+U)\log^r(1+U). \tag{6.1}$$

Putting $v = \log(1+u)$ this can be seen as vanishing of the term Δv.

For reactions B more general than being a function only of the solution u, we mention three particular examples. A description of the results corresponding to these examples would enlarge unnecessarily this work. See the references below.

- nonhomogenous reaction, $B = a(x,t)u^p$. See [20, 51, 59, 60];
- gradient dependence, $B = u^p + b\cdot\nabla(u^q)$, or $B = u^p + \mu|\nabla u|^q$. See [2, 13, 63, 66, 67];
- nonlocal reaction $B = \left(\int_\Omega u^q\right)^{p/q}$. See the survey [66].

7. Quasilinear equations

We present in this section some problems more general than (2.8), but closely related to it. They have the form (2.4), in which the operator Δ is replaced by some nonlinear diffusion operator div$\mathcal{A}$. Perhaps the first generalization of the heat equation is that obtained by taking into account a possible temperature-dependent diffusion coefficient $\psi(u)$. This leads to the quasilinear reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \mathrm{div}(\psi(u)\nabla u) + f(u). \tag{7.1}$$

A particular case is

$$\frac{\partial u}{\partial t} = \Delta u^m + u^p, \tag{7.2}$$

which is called the *porous medium equation* with reaction. Another generalization consists in taking $\mathcal{A} = \chi(\nabla u)$. For instance considering the function $\chi(z) = |z|^{\sigma-2}z$ we get the equation associated to the *σ-laplacian* operator

$$\frac{\partial u}{\partial t} = \mathrm{div}(|\nabla u|^{\sigma-2}\nabla u) + u^p. \tag{7.3}$$

Both equations, (7.2) and (7.3), have a lot of similarities: see for instance [68] for a comparison of the evolution equations without reaction. There is even an operator involving both types of nonlinearities, $\mathrm{div}(|\nabla u^m|^{\sigma-2}\nabla u^m)$, see [18]. We focus here on the first example, (7.2). A good reference for blow-up in models (7.2) and (7.3) is [64]. Much as in the previous section, different reactions can be added to these equations. We refrain from doing it here.

7.1. **The Porous Medium Equation.** Before getting into the blow-up problem for equation (7.2), and as we did for the heat equation, we review some basic facts about the porous medium equation (without reaction). Standard references are [3, 7, 8, 69], see also the recent book [71]. Our interest lies in the *slow diffusion* case, $m > 1$, in opposition to linear diffusion $m = 1$ and fast diffusion $m < 1$. Thus we consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u^m & \text{for } x \in \Omega,\ t > 0, \\ u = 0 & \text{for } x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{7.4}$$

where Ω is a bounded regular domain or else $\Omega = \mathbb{R}^N$. As to the applications, this problem models heat propagation with temperature-dependent thermal conductivity, as we have mentioned. It also models the flow of an ideal gas in a homogeneous porous medium from a macroscopic point of view, from where the name is taken. In this context u is the density of the gas.

The properties of the solutions that we want to address here are:

- *Finite speed of propagation.* The main difference between the case $m > 1$ and the case $m \le 1$ is the finite speed of propagation. This means that, if for instance the initial value has compact support, then the solution has compact support for every fixed time. The boundary of the support is called the *interface* or *free boundary.*

- *Source-type solutions.* The role of gaussians in the heat equation is played here by the *Barenblatt solutions*

$$B(x,t;M) = t^{-\sigma}\Big(C(M) - b|x|^2 t^{-2\sigma/N}\Big)_+^{1/(m-1)} \tag{7.5}$$

with $\sigma = (m - 1 + 2/N)^{-1}$, $b = \sigma(m - 1)/2mN$, and $C(M)$ is a constant depending on the mass $M = \int_{\mathbb{R}^n} B(x, t)\, dx$.

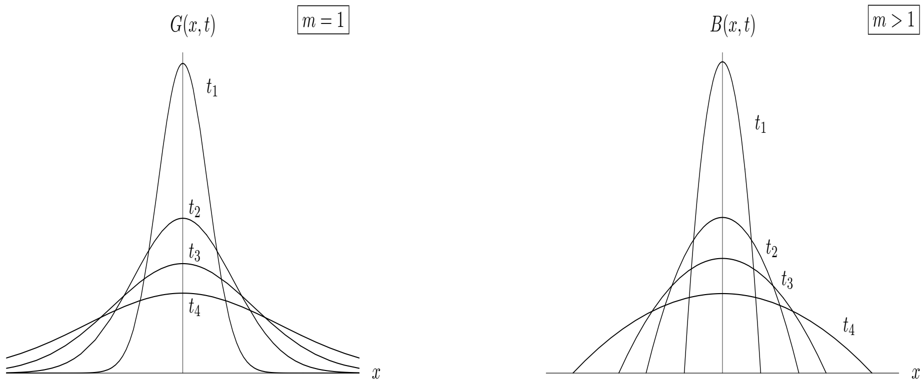

FIGURE 2. Infinite vs. finite speed of propagation: Gauss kernel and Barenblatt solution

- *Weak solution.* We observe that the above function is not regular at the interface, $B^m \notin C^2$. This is a consequence of the degenerate diffusion coefficient. In particular this means that no classical solution exist if it vanishes somewhere, and the concept of weak solution is needed.

Definition 7.1. *We say that a locally integrable function u is a weak solution to problem* (7.4) *if for each fixed* $t_1 > 0$, $u \in L^1(\Omega \times (0, t_1))$, $u^m \in L^1(0, t_1 : W_0^{1,1}(\Omega))$ *and it satisfies*

$$\int_0^{t_1}\!\!\int_\Omega \Big(u\frac{\partial\varphi}{\partial t} - \nabla(u^m)\cdot\nabla\varphi\Big)\, dxdt + \int_\Omega u_0(x)\varphi(x, 0)\, dx = 0$$

for every $\varphi \in C^1(\Omega \times (0, t_1)) \bigcap C(\overline{\Omega \times (0, t_1)})$ *vanishing on* $\partial\Omega$ *and at* $t = t_1$.

- *Maximum principle* and *conservation of mass* (when $\Omega = \mathbb{R}^N$) hold like in the heat equation.

- *Regularizing effect* is true only in the positivity set. The decay rate is $\|u(\cdot, t)\|_\infty \le ct^{-\sigma}\|u_0\|_1$ when $\Omega = \mathbb{R}^N$, or $\|u(\cdot, t)\|_\infty \le ct^{-1/(m-1)}\|u_0\|_1$ when Ω is bounded.

7.2. **Blow-up for the Porous Medium Equation with reaction.** We now pass to present the results on blow-up for the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u^m + u^p & \text{for } x \in \Omega,\ t > 0, \\ u = 0 & \text{for } x \in \partial\Omega,\ t > 0, \\ u(x, 0) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{7.6}$$

with $m > 1$, $p > 0$. As to initial data we assume $u_0 \ge 0$. See mainly [64].

In the case of bounded Ω the critical existence exponent for this problem is $p_0 = m$.

Theorem 7.1. i) *If* $p < m$ *the solution is global;*

ii) *if* $p > m$ *the solution can be global or can blow up in finite time depending on the initial datum;*

iii) *if* $p = m$ *the solution can be global or can blow up in finite time depending on the size of* Ω. *In particular it blows up if* $\lambda_1 < 1$, *it is bounded if* $\lambda_1 \ge 1$.

We recall that λ_1 is the first eigenvalue of $-\Delta$ in Ω. The argument of Kaplan works with $J(t) = \left(\int_\Omega u^m \varphi_1\right)^{1/m}$ if $p > m$, and also if $p = m$ but with $\lambda_1 < 1$. In fact J is a supersolution to the ODE $\widetilde{J}' = -\lambda_1 \widetilde{J}^m + \widetilde{J}^p$. On the other hand, there exist stationary supersolutions if $p < m$ or $p = m$ and $\lambda_1 \geq 1$.

In the case of $\mathbb{R}^N$ the critical existence exponent for problem (7.6) is $p_0 = 1$, while the Fujita exponent is $p_c = m + 2/N$. We assume u_0 vanishing at infinity, or even more, with compact support.

Theorem 7.2. i) *If* $p \leq 1$ *the solution is global;*

ii) *if* $1 < p \leq m + 2/N$ *the solution blows up in finite time;*

iii) *if* $p > m + 2/N$ *the solution can be global or can blow up in finite time depending on the initial value.*

The proof uses comparison with subsolutions and supersolutions in self-similar form

$$\underline{u}(x,t) = (t + t_0)^\alpha f(x(t + t_0)^\beta),$$

$$\overline{u}(x,t) = (T - t)^{-\alpha} F(x(T - t)^{-\beta}),$$

where the similarity exponents are

$$\alpha = \frac{1}{p-1}, \qquad \beta = \frac{p-m}{2(p-1)}. \tag{7.7}$$

In the range $m < p < m+2/N$ it also works Levine's energy method, since the Barenblatt solutions have negative energy for large t.

The best references on critical exponents are [15, 48] and of course [64]. Recall also the argument in [34] to get p_c formally: equate the reaction growth $a_r = 1/(p-1)$ to the diffusion decay $a_d = \sigma = (m - 1 + 2/N)^{-1}$.

As to the rate at which the solutions blow up, we have the natural rate for subcritical exponents.

Theorem 7.3. *Let* Ω *be a bounded convex domain in* $\mathbb{R}^N$*, or else* $\Omega = \mathbb{R}^N$*. Then*

$$\|u(\cdot,t)\|_\infty \leq c(T - t)^{-1/(p-1)} \tag{7.8}$$

provided $1 < p < m(N+2)/(N-2)$ *or* $N \leq 2$.

Like in the semilinear case, we rescale accordingly to these rate, thus looking for stabilization to a stationary solution in new variables. The asymptotic behaviour is in this way given by a self-similar solution. From now on we restrict ourselves to dimension $N = 1$. It is not only a question of simplification. There still exist difficult open questions for the multidimensional case $N > 1$. We put

$$u(x,t) = (T - t)^{-\alpha} g(\xi,\tau), \qquad \xi = (T - t)^{-\beta} x, \ \tau = \log(1 - t/T), \tag{7.9}$$

with α and β as in (7.7). The rescaled equation is then

$$\frac{\partial g}{\partial \tau} = \frac{\partial^2(g^m)}{\partial \xi^2} - \beta\xi\frac{\partial g}{\partial \xi} + g^p - \alpha g. \tag{7.10}$$

Convergence to a stationary solution G implies

$$\lim_{t\to T}(T - t)^\alpha u(\xi(T - t)^\beta, t) = \lim_{\tau\to\infty} g(\xi,\tau) = G(\xi). \tag{7.11}$$

Clearly, it is then of primary importance to classify the stationary solutions G, i.e. the possible limit self-similar profiles. This is done in [31]. Uniqueness in the case $p > m$ is still an open problem.

Theorem 7.4. i) *If $p < m$ there exists a unique profile G and it has compact support;*
ii) *if $p = m$ the unique profile is explicit*

$$G(\xi) = \begin{cases} C(\cos B\xi)^{2/(m-1)} & \text{for } |\xi| \le \pi/2B, \\ 0 & \text{elsewhere,} \end{cases}$$

where $C^{m-1} = 2m/(m^2-1)$, $B = (m-1)/2m$;
iii) *if $p > m$ there exists a profile G which is positive and behaves like $G(\xi) \sim \xi^{-2/(p-m)}$ as $\xi \to \infty$.*

The proof of convergence now relies on the construction of a Lyapunov functional for g. It is explicit if $p = m$ ($\beta = 0$),

$$\mathcal{L}_g(\tau) = \int_{\mathbb{R}} \Big(\frac{1}{2}\Big|\frac{\partial g^m}{\partial \xi}\Big|^2 - \frac{m}{p+m} g^{p+m} + \frac{\alpha m}{m+1} g^{m+1}\Big).$$

In the general case it is not an easy task to construct it, see for instance [29, 75].

Once we know the asymptotic profile, we describe the set of points where u blows up, the blow-up set $B(u)$. If the initial data is symmetric and nonincreasing for $x > 0$, there only exist three possibilities for $B(u)$: single-point blow-up, $B(u) = \{0\}$, like in the semilinear case with a power; regional blow-up, $B(u) = [-\xi_0, \xi_0]$ or global blow-up, $B(u) = \mathbb{R}$, like in the semilinear case with a logarithm. Here the localization exponent is $p = m$.

Theorem 7.5. *In the previous hypotheses, we have*

$$\begin{array}{rrl} i) & 1 < p < m & \Rightarrow\ B(u) = \mathbb{R}, \\ ii) & p = m & \Rightarrow\ B(u) = [-\xi_0, \xi_0], \qquad \xi_0 = m\pi/(m-1), \\ iii) & p > m & \Rightarrow\ B(u) = \{x = 0\}. \end{array}$$

Single-point blow-up is deduced from an estimate of the form

$$u(x,t) \le c|x|^{-2/(p-m)}, \tag{7.12}$$

see [29]. When $1 < p < m$, it is just the convergence to the self-similar profile what gives global blow-up. On the other hand, the convergence to the explicit profile in the case $p = m$ is not enough to get the desired result, since it only means $B(u) \supseteq [-\xi_0, \xi_0]$. Here we have to check that, away of the interval $[-\xi_0, \xi_0]$, the rescaled function g, solution to equation (7.10), satisfies, like any solution to the ODE $g' = -\alpha g$, the estimate $g(\xi,\tau) \le ce^{-\alpha\tau}$, see [14]. In original variables this means $u(x,t) \le c$, and thus $B(u) = [-\xi_0, \xi_0]$.

Finally, the growth of the interface can also be characterized from similarity. If u_0 has compact support, so has u at every time. Let $\operatorname{supp}(u(\cdot,t)) = [s_1(t), s_2(t)]$. Without reaction it is known that $s_1(t)$ and $s_2(t)$ grow like a power. When blow-up takes place we have

Theorem 7.6. *The interfaces are globally bounded for every $0 \le t < T$ if and only if $p \ge m$. Moreover, if $1 < p < m$ ($\beta < 0$) then*

$$|s_i(t)| \sim (T-t)^{\beta} \qquad \text{as } t \nearrow T.$$

Intersection comparison plays a crucial role in the proof; see e.g. [64].

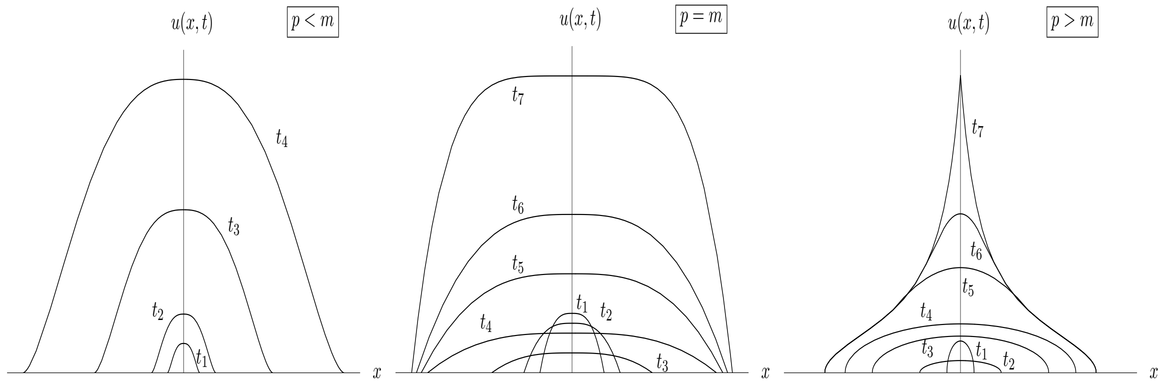

FIGURE 3. Comparison of the growth of the solution and the interface for different values of p and m

8. Problems with boundary flux

Blow-up can be produced not only by an interior reaction, but sometimes it is the effect of some flux condition on the boundary of the domain; see [22]. Let us consider for instance the following problem posed in the half-line

$$\begin{cases} u_t = u_{xx} & \text{for } x > 0,\ t \in (0,T), \\ -u_x(0,t) = u^q(0,t) & \text{for } t \in (0,T), \\ u(x,0) = u_0(x) & \text{for } x > 0. \end{cases} \tag{8.1}$$

(We write here u_t to denote partial derivative $\dfrac{\partial u}{\partial t}$, and so on). Most of the techniques used in the previous sections can be applied to this problem. In the paper [33] the critical exponents for problem (8.1) are obtained, as well as for the corresponding problems with nonlinear diffusion, porous medium or σ-laplacian. Let us then look at the problem

$$\begin{cases} u_t = (u^m)_{xx} & \text{for } x > 0,\ t \in (0,T), \\ -(u^m)_x(0,t) = u^q(0,t) & \text{for } t \in (0,T), \\ u(x,0) = u_0(x) & \text{for } x > 0, \end{cases} \tag{8.2}$$

with $m \geq 1$. For this problem it is proved

Theorem 8.1. i) *There exist blow-up solutions if and only if* $q > (m+1)/2$;
ii) *if* $(m+1)/2 < q \leq m+1$ *all the solutions blow up, whereas if* $q > m+1$ *there exist blow-up solutions as well as global solutions;*
iii) *the interface is globally bounded if and only if* $q \geq m$;
iv) *the solutions that blow up at time* T *behave at* $x = 0$ *like* $(T-t)^{-1/(2q-m-1)}$.

See also [19, 24]. We compare in Fig. 4 the above numbers with the exponents obtained for the semilinear problem (3.1) in dimension $N = 1$. We call s_0 = global existence exponent; s_c = Fujita exponent; s_ℓ = localization exponent; α = blow-up rate.

The general problem associated with blow-up produced by a boundary flux in several variables, posed in a bounded domain, in the exterior of a bounded domain or in a half-space, has also been considered by several authors, but only partial answers have been obtained. We mention instead a mixed problem in one dimension in which blow-up can be produced by an interior reaction or a boundary flux, or the combination of the two.

problem\exponent	s_0	s_c	s_ℓ	α
reaction (3.1)	1	$m+2$	m	$\dfrac{1}{p-1}$
flux (8.2)	$\dfrac{m+1}{2}$	$m+1$	m	$\dfrac{1}{2q-m-1}$

Figure 4. Critical exponents for problems (3.1) and (8.2)

In [57] the following problem is studied

$$\begin{cases} u_t = (u^m)_{xx} + u^p & \text{for } x > 0,\ t \in (0,T), \\ -(u^m)_x(0,t) = u^q(0,t) & \text{for } t \in (0,T), \\ u(x,0) = u_0(x) & \text{for } x > 0. \end{cases} \tag{8.3}$$

Here the critical exponents become *critial curves* in pq–plane.

Theorem 8.2. i) *The global existence region is* $\{0 < p \le 1,\ 0 < q \le (m+1)/2\}$*;*
ii) *the Fujita curve is* $\{p = m+2,\ q \ge m+1\} \cup \{q = m+1,\ p \ge m+2\}$*;*
iii) *the localization curve is* $\max\{p,\ q\} = m$*;*
iv) *the blow-up rate is* $\alpha = \min\{1/(p-1),\ 1/(2q-m-1)\}$.

It is interesting to compare the effect of both reactive terms in this problem, in particular which one is responsable for blow-up. This becomes clear after the theorem: above or below the critical line $m+p = 2q$ the exponents are given by Fig. 4. Even more, when $q \le (m+1)/2$ and $p > m+2$ the reaction term by itself is not able to produce blow-up of every solution; it is the combined effect of flux and reaction what is needed: the boundary flux makes the solution so large that reaction makes it blow-up. Similar behaviour occurs when $p \le 1$ and $q > m+1$. On the other hand, a phenomenon of asymptotic simplification takes also place: the asymptotic behaviour of blow-up solutions to problem (8.3) is given by a self-similar function, solution to problem (3.1) if $m+p > 2q$, to problem (8.2) if $m+p < 2q$, or to the full problem (8.3) if $m+p = 2q$.

We finally present another example, described in [20], that can be considered as an intermediate problem between (3.1) and (8.2). If $a(x)$ is a compactly supported function, for instance a characteristic function $a(x) = \chi_{[-L,L]}$, then the critical exponents for the problem

$$\begin{cases} u_t = (u^m)_{xx} + a(x)u^p & \text{for } x \in \mathbb{R},\ t \in (0,T), \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{8.4}$$

are those of problem (8.2), instead of problem (3.1). And this happens even for L large. On the other hand, the blow-up rates and the blow-up sets are the corresponding to one problem or to the other depending on wether $p > m$ (problem (3.1)), or $p < m$ (problem (8.2)). A similar asymptotic simplification as above also occurs for this problem: the asymptotic behaviour is given by a self-similar function, solution to problem (3.1) if $p > m$, to problem (8.2) if $p < m$, or to the original problem (8.4) if $p = m$.

9. Systems. Critical exponents curves

Things become more complicated in the case of systems: only partial answers are known. We mention here some results concerning the critical exponents and the blow-up rates. Thus consider the simplest system

$$\begin{cases} u_t = \Delta u + v^p & \text{for } x \in \Omega,\ 0 < t < T, \\ v_t = \Delta v + u^q & \text{for } x \in \Omega,\ 0 < t < T, \\ u = v = 0 & \text{for } x \in \partial\Omega,\ 0 < t < T, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \\ v(x,0) = v_0(x) & \text{for } x \in \Omega. \end{cases} \tag{9.5}$$

When Ω is bounded and $p,\ q \geq 1$, we can apply Kaplan's technique with the functions $J = \int_\Omega u\varphi_1$, $K = \int_\Omega v\varphi_1$, where φ_1 is the first eigenfunction of $-\Delta$ in Ω. Thus we have $J \geq \widetilde{J}$, $K \geq \widetilde{K}$, where $\widetilde{J}$ and $\widetilde{K}$ satisfy the ODE system

$$\begin{cases} \widetilde{J}' = -\lambda_1 \widetilde{J} + \widetilde{K}^p, \\ \widetilde{K}' = -\lambda_1 \widetilde{K} + \widetilde{J}^q, \end{cases} \tag{9.6}$$

λ_1 being the eigenvalue associated to φ_1.

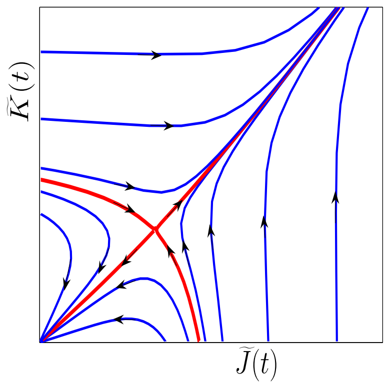

FIGURE 5. Flow for system (9.6) in the case $pq > 1$.

This system leads easily to blow-up provided $pq > 1$ and the initial values $J_0 = \int_\Omega u_0\varphi_1$ and $K_0 = \int_\Omega v_0\varphi_1$ are large, see [17]. In fact, as it is shown in Fig. 5, if (J_0, K_0) lies above the stable manifold, then the corresponding trajectory escapes to infinity tangent to the unstable manifold. This means $\widetilde{K} \sim \widetilde{J}^{(q+1)/(p+1)}$. Thus $\widetilde{J}' \sim \widetilde{J}^{p(q+1)/(p+1)}$, and since the exponent is $p(q+1)/(p+1) > 1$ whenever $pq > 1$, we have that $\widetilde{J}$ and $\widetilde{K}$ (and also J and K) go to infinity in finite time.

Analogous argument gives blow-up for the quasilinear system

$$\begin{cases} u_t = \Delta u^m + v^p & \text{for } x \in \Omega,\ 0 < t < T, \\ v_t = \Delta v^n + u^q & \text{for } x \in \Omega,\ 0 < t < T, \\ u = v = 0 & \text{for } x \in \partial\Omega,\ 0 < t < T, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \\ v(x,0) = v_0(x) & \text{for } x \in \Omega, \end{cases} \tag{9.7}$$

in the case $p \geq n$, $q \geq m$ and $pq > mn$ or $pq = mn$ with $\lambda_1 < 1$, see [32].

As to the case $\Omega = \mathbb{R}^N$, (with $m = n = 1$), a careful iterative use of Duhamel's formula (3.4) allows Escobedo and Herrero to prove the following result; see [16].

Theorem 9.1. *Related to system* (9.5) *with* $\Omega = \mathbb{R}^N$, *it holds*

i) *the critical existence curve is* $pq = 1$, *i.e. there exist blow-up solutions if and only if* $pq > 1$;

ii) *the Fujita curve is* $pq = \gamma \equiv 1 + \frac{2}{N}(\max\{p, q\} + 1)$*, i.e. if* $1 < pq \le \gamma$ *then every solution blows up, while if* $pq > \gamma$ *the solution blows up only if the initial values are large.*

For the quasilinear system (9.7) in $\mathbb{R}^N$ the conjecture is the following:

- the critical existence curve is $pq = mn$;
- the Fujita curve is $pq = mn + \frac{2}{N}\max\{p + n, q + m\}$.

It has been proved only for $0 < m = n < 1$ by Qi and Levine, [61]. It remains open in the general case.

Looking now at the blow-up rates, the natural ones are obtained solving the ODE which results dropping the diffusions:

$$U' = V^p, \qquad V' = U^q. \tag{9.8}$$

This gives

$$U(t) = c_1(T - t)^{-(p+1)/(pq-1)}, \qquad V(t) = c_2(T - t)^{-(q+1)/(pq-1)}. \tag{9.9}$$

The fact that solutions to system (9.5) satisfy the rates (9.9) (of course with some restrictions on the exponents) has been proved in [10]; see also [1].

More general systems have also been treated in the literature, including other diffusion operators, other reaction functions (even nonlocal) depending on both variables, or systems of equations coupled through some flux boundary condition. The answers are not yet as complete as for the corresponding single equations. We refer to [15] just to show some examples.

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