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A NEW PROOF OF THE CONVERGENCE OF DISTRIBUTED OPTIMAL CONTROLS ON THE INTERNAL ENERGY IN MIXED ELLIPTIC PROBLEMS *

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Abstract

We consider two steady-state heat conduction problems P and P_α (for each $\alpha > 0$) with mixed boundary conditions for the same Poisson equation. The difference between both problems is that on the boundary portion Γ_1 a Dirichlet condition is verified for P and a Newton condition with transfer coefficient α is verified for P_α. We formulate distributed optimal control problems, for a suitable cost function, over the internal energy g in the material. We make a new proof with respect to the one given in C.M. Gariboldi - D.A. Tarzia, App. Math. Optim. 47 (2003), 213-230 on the strongly convergence when $\alpha \to \infty$ of the optimal control g_{op_α}, the system state $u_{g_{op_\alpha}\alpha}$ and the adjoint state $p_{g_{op_\alpha}\alpha}$ to the optimal control g_{op}, system state $u_{g_{op}}$ and adjoint state $p_{g_{op}}$ corresponding to P_α and P respectively. For this proof we eliminate the restriction on the constant of coerciveness of the bilinear form, and we use properties of the cost function and the theory of variational equalities instead of the fixed point theorem.

Key words: Variational Inequality, Distributed Optimal Control, Mixed Elliptic Problem, Adjoint State, Steady-State Stefan Problem, Optimality Condition.
AMS Subject Classifications: 49J20, 35J85, 35R35.

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1 Introduction

We consider a bounded domain Ω in $\mathbb{R}^n$ whose regular boundary Γ consists of the union of two disjoint portions Γ_1 y Γ_2 with meas$(\Gamma_1)> 0$ and meas$(\Gamma_2)> 0$. We denote with meas (Γ) the (n-1)-dimensional Lebesgue measure of Γ.

We consider the following two steady-state heat conduction problems P and P_α (for each parameter $\alpha > 0$) respectively with mixed boundary conditions:

$$-\Delta u = g \text{ in } \Omega \qquad u|_{\Gamma_1} = B \qquad -\frac{\partial u}{\partial n}|_{\Gamma_2} = q \tag{1}$$

and

$$-\Delta u = g \text{ in } \Omega \qquad -\frac{\partial u}{\partial n}|_{\Gamma_1} = \alpha(u - B) \qquad -\frac{\partial u}{\partial n}|_{\Gamma_2} = q \tag{2}$$

where g is the internal energy in Ω, B is the temperature on Γ_1 for (1) and the temperature of the external neighboord of Γ_1 for (2), q is the heat flux on Γ_2 and $\alpha > 0$ is the heat transfer coefficient of Γ_1 (Newton's law on Γ_1), that satisfy the following assumptions:

$$g \in H = L^2(\Omega)\,, \qquad q \in L^2(\Gamma_2)\,, \qquad B \in H^{\frac{1}{2}}(\Gamma_1). \tag{3}$$

Problems (1) and (2) can be considered as the steady-state Stefan problem for suitable data q, g and B [5], [8], [11], [18], [19] and [21] .

Let u_g and $u_{g\alpha}$ be the unique solutions of the mixed elliptic problems (1) and (2) respectively whose variational equalities are given by [15], [19]:

$$a(u_g, v) = L_g(v), \ \forall v \in V_0, \ u_g \in K \tag{4}$$

and

$$a_\alpha(u_{g\alpha}, v) = L_{g\alpha}(v), \ \forall v \in V, \ u_{g\alpha} \in V \tag{5}$$

where

$$V = H^1(\Omega)\,; \qquad V_0 = \{v \in V/\, v|_{\Gamma_1} = 0\}\,;$$

$$K = v_0 + V_0; \qquad (g,h) = \ (g,h)_H = \int_\Omega gh\, dx; \tag{6}$$

$$a(u,v) = \int_\Omega \nabla u.\nabla v \ dx \ ; \qquad a_\alpha(u,v) = a(u,v) + \alpha \int_{\Gamma_1} Bv \ d\gamma$$

$$L_g(v) = (g,v)_H - \int_{\Gamma_2} qv\, d\gamma\,; \quad L_{g\alpha}(v) = \ L_g(v) + \alpha \int_{\Gamma_1} Bv \ d\gamma$$

for a given $v_0 \in V$, $v_0|_{\Gamma_1} = B$.

We consider g as a control variable for the cost functionals $J : H \to \mathbb{R}_0^+$ and $J_\alpha : H \to \mathbb{R}_0^+$ respectively given by:

$$J(g) = \frac{1}{2} \|u_g - z_d\|_H^2 + \frac{M}{2} \|g\|_H^2 \tag{7}$$

and

$$J_\alpha(g) = \frac{1}{2} \|u_{g\alpha} - z_d\|_H^2 + \frac{M}{2} \|g\|_H^2 \tag{8}$$

where $z_d \in H$ is given and $M =$const.> 0.

Then we can formulate the following distributed optimal control problems [7], [9], [10], [12] and [16]:

$$\text{Find } g_{op} \in H \text{ such that } J(g_{op}) = \min_{g \in H} J(g) \tag{9}$$

and

$$\text{Find } g_{op_\alpha} \in H \text{ such that } J_\alpha(g_{op_\alpha}) = \min_{g \in H} J_\alpha(g) \tag{10}$$

respectively.

The use of the variational inequality theory in connection with optimal control problems was done, for example, in [1], [2], [3], [4], [6], [14] and [17]. In [13] an optimization problem corresponding to (1) is studied in order to avoid a change phase process.

In Section 2 we get that the functional J is coercive and Gâteaux differentiable on H, J' is a lipschitzian and strictly monotone application on H. We also obtain the existence and uniqueness of the distributed optimal control problem (9). Similary, in Section 3 we get that the functional J_α is coercive and Gâteaux differentiable on H, J'_α is a lipschitzian and strictly monotone application on H for all $\alpha > 0$. We also obtain the existence and uniqueness of the distributed optimal control problem (10) and strongly convergence (when $\alpha \to \infty$) of the states system (2) and the corresponding adjoint states to the respectives of the system (1), for all $g \in H$. Sections 2 and 3 follow [12].

In Section 4 we study the convergence when $\alpha \to \infty$ of the optimal control problem (10) corresponding to the state system (2). We prove that the optimal state system $u_{g_{op_\alpha}\alpha}$ and the optimal adjoint system $p_{g_{op_\alpha}\alpha}$ of problem (10) are strongly convergent in V to the corresponding $u_{g_{op}}$ and $p_{g_{op}}$ for problem (9) respectively when $\alpha \to \infty$. Finally the strong convergence in H of the optimal control g_{op_α} of problem (10) to the optimal control g_{op} of problem (9) is also proved when $\alpha \to \infty$. This proof is new with respect to the one given in [12]. We have eliminated the restriction on the constant of coerciveness of the bilinear form a and we use the variational equality theory and the optimal control problem instead of the fixed point theory.

2 Problem P and its Corresponding Optimal Control Problem

Let $C : H \to V_0$ be the application such that:

$$C(g) = u_g - u_0 \tag{11}$$

where u_0 is the solution of problem (4) for $g = 0$ whose variational equality is given by:

$$a(u_0, v) = L_0(v), \ \forall v \in V_0, \ u_0 \in K \tag{12}$$

with

$$L_0(v) = -\int_{\Gamma_2} qv \, d\gamma.$$

Let $\Pi : H \times H \to \mathbb{R}$ and L: $H \to \mathbb{R}$ be defined by the following expressions:

$$\Pi(g, h) = (C(g), C(h)) + M(g, h), \quad \forall \ g, h \in H \tag{13}$$

$$L(g) = (C(g), z_d - u_0), \quad \forall \ g \in H.$$

We have that a is a bilinear, continuous and symmetric form on V and coercive on V_0, that is [15], [19]:

$$\exists \ \lambda > 0 \text{ such that } \ a(v, v) \geq \lambda \left\| v \right\|_V^2, \forall \, v \in V_0. \tag{14}$$

Lemma 2.1. (i) C is a linear and continuous application, Π is a linear, continuous, symmetric and coercive form on H, that is:

$$\Pi(g, g) \geq M \left\| g \right\|_H^2, \forall \, g \in H \tag{15}$$

and L is linear and continuous on H.

(ii) J can be also written as:

$$J(g) = \frac{1}{2}\Pi(g, h) - L(g) + \frac{1}{2} \left\| u_0 - z_d \right\|_H^2, \forall \, g \in H. \tag{16}$$

(iii) There exists a unique optimal control $g_{op} \in H$ such that:

$$J(g_{op}) = \min_{g \in H} \ J(g) \tag{17}$$

(iv) The application $g \in H \to u_g \in V$ is lipschitzian, that is:

$$\left\| u_{g_2} - u_{g_1} \right\|_V \leq \frac{1}{\lambda} \left\| g_2 - g_1 \right\|_H, \forall \, g_1, g_2 \in H. \tag{18}$$

(v) J is a Gâteaux differentiable functional and J' is given by:

$$\langle J'(g), h \rangle = (u_g - z_d, C(h)) + M(g, h) = \Pi(g, h) - L(g) \, , \quad \forall \ g, h \in H \, . \tag{19}$$

(vi) The Gâteaux derivative of J can be written as:

$$J'(g) = p_g + Mg, \forall \, g \in H. \tag{20}$$

where the adjoint state p_g corresponding to problem (1) or (4), for each $g \in H$, is the unique solution of the following mixed elliptic problem:

$$-\Delta p_g = u_g - z_d \ \text{ in } \Omega; \ p_g|_{\Gamma_1} = 0 \, ; \ \frac{\partial p_g}{\partial n}|_{\Gamma_2} = 0 \tag{21}$$

whose variational formulation is given by:

$$a(p_g, v) = (u_g - z_d, v), \forall v \in V_0,\ \ p_g \in V_0. \tag{22}$$

Moreover, the adjoint state p_g satisfy the following equalities:

$$(p_g, h) = (u_g - z_d, C(h)) = a(p_g, C(h))\ \ \forall\ g, h \in H\,. \tag{23}$$

(vii) The optimality condition for the problem (9) is given by $J'(g_{op}) = 0$ in H, that is:

$$p_{g_{op}} + M g_{op} = 0 \ \text{ in } H \tag{24}$$

(viii) We have the following inequality:

$$\|p_{g_2} - p_{g_1}\|_V \leq \frac{1}{\lambda} \|u_{g_2} - u_{g_1}\|_H \ \ \forall\, g_1, g_2 \in H \tag{25}$$

Proof. (i)-(iv) See [12].

(v)-(viii) The mean ideas of the proof are the following expressions:

$$\begin{aligned} a)\ \frac{1}{t}[J(g + t(f-g)) - J(g)] \ &= \ \frac{t}{2}(u_f - u_g, u_f - u_g) + (u_g - z_d, u_f - u_g) \\ &\quad + M(g, f-g) + \frac{Mt}{2}(f-g, f-g) \end{aligned}$$

$$b)\ a(p_g, C(h)) = a(p_g, u_h - u_0) = a(p_g, u_h) - a(p_g, u_0) = (p_g, h)$$

$$c)\ \lambda \|p_{g_2} - p_{g_1}\|_V^2 \leq a(p_{g_2} - p_{g_1}, p_{g_2} - p_{g_1}) \leq \|u_{g_2} - u_{g_1}\|_H \|p_{g_2} - p_{g_1}\|_H$$

and the details are given in [12]. ■

Now, we are in conditions for obtaining other properties of the functional J.

Lemma 2.2. (i) The application $g \in H \to p_g \in V_0$ is strictly monotone. Moreover, we have:

$$(p_{g_2} - p_{g_1}, g_2 - g_1) = \|u_{g_2} - u_{g_1}\|_H^2 \geq 0, \quad \forall\, g_1, g_2 \in H. \tag{26}$$

(ii) J is coercive or H-elliptic, that is:

$$(1-t)J(g_2) + tJ(g_1) - J((1-t)g_2 + tg_1) \geq \frac{Mt(1-t)}{2} \|g_2 - g_1\|_H^2\ , \ \ \forall g_1, g_2 \in H, \quad \forall\, t \in [0,1]. \tag{27}$$

(iii) J' is a Lipschitzian and strictly monotone application, that is:

$$\|J'(g_2) - J'(g_1)\|_H \leq (M + \frac{1}{\lambda^2})\ \|g_1 - g_2\|_H \tag{28}$$

and

$$\langle J'(g_2) - J'(g_1), g_2 - g_1\rangle = \|u_{g_2} - u_{g_1}\|_H^2 + M\, \|g_2 - g_1\|_H^2 \geq M\, \|g_2 - g_1\|_H^2\ , \ \ \forall g_1, g_2 \in H. \tag{29}$$

Proof. See [12] ■

3 Problem P_α and Its Corresponding Optimal Control Problem

Let $\Pi_\alpha : H \times H \to \mathbb{R}$, $L_\alpha : H \to \mathbb{R}$ and $C_\alpha : H \to V$ be defined by:

$$\Pi_\alpha(g,h) = (C_\alpha(g), C_\alpha(h)) + M(g,h)\,, \quad \forall\, g,\, h \in H$$

$$L_\alpha(g) = (C_\alpha(g), z_d - u_{0\alpha}), \quad \forall\, g \in H \tag{30}$$

$$C_\alpha(g) = u_{g\alpha} - u_{0\alpha}, \quad \forall\, g \in H$$

where $u_{g\alpha}$ is the unique solution of the variational equality (5), $u_{0\alpha}$ is the unique solution of (5) for $g = 0$ whose variational equality is given by:

$$a_\alpha(u_{0\alpha}, v) = L_{0\alpha}(v), \forall v \in V, u_{0\alpha} \in V \tag{31}$$

with

$$L_{0\alpha}(v) = \alpha \int_{\Gamma_1} Bv\; d\gamma - \int_{\Gamma_2} qv\, d\gamma \tag{32}$$

and a_α is a bilinear, continuous, symmetric and coercive form on V, that is:

$$a_\alpha(v,v) \geq \lambda_\alpha \|v\|_V^2\,, \forall v \in V. \tag{33}$$

where $\lambda_\alpha = \lambda_1 \min(1,\alpha) > 0$ for all $\alpha > 0$ and λ_1 is the coerciveness constant for the bilinear form a_1 [20].

We can obtain similaries properties to Lemma 2.1, following [13], [15], [16] and [19] which proof is omitted.

Lemma 3.1. Let $\alpha > 0$ be. (i) C_α is a linear and continuous application, Π_α is linear, continuous, symmetric and coercive on H, that is:

$$\Pi_\alpha(g,g) \geq M \|g\|_H^2\,, \forall g \in H. \tag{34}$$

and L_α is linear and continuous on H.

(ii) J_α can be also written as:

$$J_\alpha(g) = \frac{1}{2}\Pi_\alpha(g,h) - L_\alpha(g) \;+ \frac{1}{2}\|u_{0\alpha} - z_d\|_H^2\,, \;\forall\, g \in H. \tag{35}$$

(iii) There exists a unique optimal control $g_{op_\alpha} \in H$ such that:

$$J_\alpha(g_{op_\alpha}) = \min_{g \in H} J_\alpha(g). \tag{36}$$

(iv) The application $g \in H \to u_{g\alpha} \in V$ is lipschitzian, that is:

$$\|u_{g_2\alpha} - u_{g_1\alpha}\|_V \leq \frac{1}{\lambda_\alpha}\|g_2 - g_1\|_H\,, \forall g_1, g_2 \in H. \tag{37}$$

(v) J_α is Gâteaux differentiable functional and J'_α is given by:

$$\langle J'_\alpha(g), h\rangle = (u_{g\alpha} - z_d, C_\alpha(h)) + M(g,h) = \Pi_\alpha(g,h) - L_\alpha(g),\ \forall\ g,h \in H \tag{38}$$

(vi) The Gâteaux derivative of J_α can be written as:

$$J'_\alpha(g) = p_{g\alpha} + Mg, \forall\, g \in H \tag{39}$$

where the adjoint state $p_{g\alpha}$ is the unique solution of the following mixed elliptic problem corresponding to (2) or (5), for each $g \in H$ and $\alpha > 0$:

$$-\Delta p_{g\alpha} = u_{g\alpha} - z_d \text{ in } \Omega;\quad -\frac{\partial p_{g\alpha}}{\partial n}|_{\Gamma_1} = \alpha p_{g\alpha};\ \frac{\partial p_{g\alpha}}{\partial n}|_{\Gamma_2} = 0 \tag{40}$$

whose variational formulation is given by:

$$a_\alpha(p_{g\alpha}, v) = (u_{g\alpha} - z_d, v), \forall v \in V, p_{g\alpha} \in V. \tag{41}$$

where $u_{g\alpha}$ is the unique solution of (5). Moreover, the adjoint state $p_{g\alpha}$ satisfies the following equalities:

$$(p_{g\alpha}, h) = (u_{g\alpha} - z_\alpha,\, C_\alpha(h)) = a_\alpha(p_{g\alpha},\, C_\alpha(h)), \forall\, g,h \in H. \tag{42}$$

(vii) The optimality condition for problem (10) is given by $J'_\alpha(g_{op_\alpha}) = 0$ in $H,$ that is:

$$p_{g_{op_\alpha}\alpha} + Mg_{op_\alpha} = 0 \text{ in } H. \tag{43}$$

(viii) We have the following property:

$$\|p_{g_{2\alpha}} - p_{g_{1\alpha}}\|_V \le \frac{1}{\lambda_\alpha} \|u_{g_{2\alpha}} - u_{g_{1\alpha}}\|_H\ \ \forall\, g_{1\alpha}, g_{2\alpha} \in H \tag{44}$$

Proof. See [12] ■

Remark 1. We note the double dependence on the parameter α for the optimal state system $u_{g_{op_\alpha}\alpha}$ and the adjoint state $p_{g_{op_\alpha}\alpha}$.

Lemma 3.2. (i) The operator $g \in H \to p_{g\alpha} \in V$ is strictly monotone, that is:

$$(p_{g_2\alpha} - p_{g_1\alpha}, g_2 - g_1) = \|u_{g_2\alpha} - u_{g_1\alpha}\|_H^2 \ge 0, \forall\, g_1, g_2 \in H. \tag{45}$$

(ii) J_α is coercive or H-elliptic, that is:

$$(1-t)J_\alpha(g_2)+tJ_\alpha(g_1)-J_\alpha((1-t)g_2+tg_1) \ge \frac{Mt(1-t)}{2} \|g_2 - g_1\|\, a_H^2, \forall g_1, g_2 \in H; \forall t \in [0,1]. \tag{46}$$

(iii) J'_α is a Lipschitzian and strictly monotone operator, that is:

$$\|J'_\alpha(g_2) - J'_\alpha(g_1)\|_H \le (M + \frac{1}{\lambda_\alpha^2}) \|g_1 - g_2\|_H, \forall\, g_1, g_2 \in H \tag{47}$$

and

$$\langle J'_\alpha(g_2) - J'_\alpha(g_1), g_2 - g_1\rangle \ge M \|g_2 - g_1\|_H^2, \forall g_1, g_2 \in H. \tag{48}$$

Proof. See [12] ■

Now, we will prove the following result of convergence when $\alpha \to \infty$.

Lemma 3.3. For all $\alpha > 0,\ q \in L^2(\Gamma_2),\ \ B \in H^{\frac{1}{2}}(\Gamma_1)$, we have the following limits:

$$\begin{aligned} \text{i)}\ \lim_{\alpha\to\infty} \|u_{g\alpha} - u_g\|_V &= 0, \forall g \in H \\ \text{ii)}\ \lim_{\alpha\to\infty} \|u_{0\alpha} - u_0\|_V &= 0 \\ \text{iii)}\ \lim_{\alpha\to\infty} \|p_{g\alpha} - p_g\|_V &= 0, \forall g \in H. \end{aligned} \tag{49}$$

Proof. (i) Taking into account [12] and following [19] and [20] we obtain that there exists C_1, a constant independient of α, such that for large α:

$$\|u_{g\alpha} - u_g\|_V^2 \leq \frac{C_1}{\lambda_1} \qquad , \qquad (\alpha - 1)\int_{\Gamma_1} (u_{g\alpha} - u_g)^2 d\gamma \leq \frac{(C_1)^2}{\lambda_1} \tag{50}$$

and we deduce that there exists $w_g \in K$ such that:

$$a(w_g, v) = L_g(v), \forall v \in V_0, w_g \in K \tag{51}$$

and by uniqueness, we have $w_g = u_g$. Therefore, $u_{g\alpha} \to u_g$ strongly in V as $\alpha \to \infty$ because the following inequality:

$$\lambda_1 \|u_{g\alpha} - u_g\|_V^2 \leq L_g(u_{g\alpha} - u_g) - a(u_{g\alpha}, u_{g\alpha} - u_g).$$

For the case (ii) we take $g = 0$ in the case (i).

(iii) We prove that there exists C_2 a constant independent of α, for large α, such that:

$$\|p_{g\alpha} - p_g\|_V^2 \leq \frac{C_2}{\lambda_1} \qquad , \qquad (\alpha - 1)\int_{\Gamma_1} (p_{g\alpha} - p_g)^2 d\gamma \leq \frac{(C_2)^2}{\lambda_1} \tag{52}$$

and we deduce that there exists $\xi_g \in V_0$ such that:

$$a(\xi_g, v) = (u_g - z_d, v), \forall v \in V_0, \xi_g \in V_0. \tag{53}$$

and by uniqueness, we obtain $\xi_g = p_g$. Therefore, taking into account the following inequality:

$$\lambda_1 \|p_{g\alpha} - p_g\|_V^2 \leq (u_{g\alpha} - z_d, p_{g\alpha} - p_g) - a(p_g, p_{g\alpha} - p_g)$$

we get that $p_{g\alpha} \to p_g$ strongly in V. ■

4 Convergence of Problem $\mathbf{P}_\alpha$ and its Corresponding Optimal Control as $\alpha \to \infty$

In this section we will make a new proof with respect to the one given in [12] of the strongly convergence of the optimal control g_{op_α} of problem (10) and its corresponding adjoint state $p_{g_{op_\alpha}\alpha}$ (41) to the optimal control g_{op} of problem (9) and its corresponding adjoint state $p_{g_{op}}$ (22) respectively when the parameter α (heat transfer coefficient on Γ_1) goes to infinity. We will eliminate the restriction on the constant of coerciveness of the bilinear form a and we will use properties of the cost function and the variational equality theory instead of the fixed point theorem.

Theorem 4.1. (i) If $p_{g_{op}}$ and $p_{g_{op_\alpha}\alpha}$ are the corresponding adjoint state of the problem (9) and problem (10) respectively, then:

$$\lim_{\alpha\to\infty} \left\| p_{g_{op_\alpha}\alpha} - p_{g_{op}} \right\|_V = 0 \tag{54}$$

(ii) If g_{op} and g_{op_α} are the solutions of the problem (9) and problem (10) respectively, then:

$$\lim_{\alpha\to\infty} \left\| g_{op_\alpha} - g_{op} \right\|_H = 0. \tag{55}$$

(iii) If $u_{g_{op}}$ and $u_{g_{op_\alpha}\alpha}$ are the corresponding solutions of the problem P and problem P_α respectively, then:

$$\lim_{\alpha\to\infty} \left\| u_{g_{op_\alpha}\alpha} - u_{g_{op}} \right\|_V = 0. \tag{56}$$

Proof. First we will prove some preliminary results. Since g_{op_α} is the solution of the problem (10), we have the following inequality:

$$\frac{1}{2}\left\| u_{g_{op_\alpha}\alpha} - z_d \right\|_H^2 + \frac{M}{2}\left\| g_{op_\alpha} \right\|_H^2 \leq \frac{1}{2}\left\| u_{g\alpha} - z_d \right\|_H^2 + \frac{M}{2}\left\| g \right\|_H^2, \ \forall g \in H,$$

then, taking $g = 0$, we have:

$$\frac{1}{2}\left\| u_{g_{op_\alpha}\alpha} - z_d \right\|_H^2 + \frac{M}{2}\left\| g_{op_\alpha} \right\|_H^2 \leq \frac{1}{2}\left\| u_{0\alpha} - z_d \right\|_H^2 \leq C_3, \forall \alpha > 0,$$

where C_3 is a constant independent of parameter α because $u_{0\alpha}$ is convergent when $\alpha \to \infty$. Therefore

$$\left\| g_{op_\alpha} \right\|_H \leq C_4 \text{ and } \left\| u_{g_{op_\alpha}\alpha} \right\|_H \leq C_5 \tag{57}$$

where C_4 and C_5 are constants independent of α. Now, if we take $v = u_{g_{op_\alpha}\alpha} - u_{g_{op}}$ in the variational equality (5), following [19] and [20] we obtain for $\alpha > 1$:

$$\begin{aligned} \lambda_1 \left\| u_{g_{op_\alpha}\alpha} - u_{g_{op}} \right\|_V^2 + (\alpha-1)\int_{\Gamma_1} (u_{g_{op_\alpha}\alpha} - u_{g_{op}})^2 d\gamma &\leq a_\alpha(u_{g_{op_\alpha}\alpha} - u_{g_{op}}, u_{g_{op_\alpha}\alpha} - u_{g_{op}}) \\ &\leq C_6 \left\| u_{g_{op_\alpha}\alpha} - u_{g_{op}} \right\|_V \end{aligned}$$

where $C_6 = C_6(g_{op}, q, u_{g_{op}})$ is independent of α. Next, we have:

$$\left\| u_{g_{op_\alpha}\alpha} - u_{g_{op}} \right\|_V^2 \leq \frac{C_6}{\lambda_1} \quad , \quad (\alpha-1)\int_{\Gamma_1} (u_{g_{op_\alpha}\alpha} - u_{g_{op}})^2 d\gamma \leq \frac{(C_6)^2}{\lambda_1} \tag{58}$$

and therefore we deduce that:

$$\exists\, \eta \in V \text{ sucht that } u_{g_{op_\alpha}\alpha} \rightharpoonup \eta \text{ weakly in } V, \tag{59}$$

and because the following inequalities:

$$0 \leq \int_{\Gamma_1} (\eta - u_{g_{op}})^2 d\gamma \leq \liminf_{\alpha\to\infty} \int_{\Gamma_1} (u_{g_{op_\alpha}\alpha} - u_{g_{op}})^2 d\gamma = 0,$$

we obtain that $\eta \in K$. Next, if we take $v = p_{g_{op_\alpha}\alpha} - p_{g_{op}}$ in the variational equality (41) we get:

$$\lambda_1 \left\|p_{g_{op_\alpha}\alpha} - p_{g_{op}}\right\|_V^2 + (\alpha - 1)\int_{\Gamma_1} (p_{g_{op_\alpha}\alpha} - p_{g_{op}})^2 d\gamma \leq a_\alpha(p_{g_{op_\alpha}\alpha} - p_{g_{op}}, p_{g_{op_\alpha}\alpha} - p_{g_{op}}) \leq C_7 \left\|p_{g_{op_\alpha}\alpha} - p_{g_{op}}\right\|_V,$$

with $C_7 = C_7(C_5, p_{g_{op}})$. Next, we obtain:

$$\left\|p_{g_{op_\alpha}\alpha} - p_{g_{op}}\right\|_V^2 \leq \frac{C_7}{\lambda_1} \quad , \quad (\alpha - 1)\int_{\Gamma_1} (p_{g_{op_\alpha}\alpha} - p_{g_{op}})^2 d\gamma \leq \frac{(C_7)^2}{\lambda_1} \tag{60}$$

and therefore we deduce that:

$$\exists \xi \in V \text{such that } p_{g_{op_\alpha}\alpha} \rightharpoonup \xi \text{ weakly in } V \tag{61}$$

and by the following inequality:

$$0 \leq \int_{\Gamma_1} (\xi - p_{g_{op}})^2 d\gamma \leq \liminf_{\alpha\to\infty} \int_{\Gamma_1} (p_{g_{op_\alpha}\alpha} - p_{g_{op}})^2 d\gamma = 0$$

we obtain $\xi \in V_0$. Now, we consider $v \in V_0$ and taking into acount (59) and (61), from the variational equality (41) we have:

$$a(\xi, v) = (\eta - z_d, v), \forall v \in V_0, \xi \in V_0. \tag{62}$$

Next, from (57) we deduce that there exists $f \in H$ such that $g_{op_\alpha} \rightharpoonup f$ weakly in H. Therefore if we put $v \in V_0$ in the variational equality (5) and we pass to the limit $\alpha \to \infty$, we obtain:

$$a(\eta, v) = (f, v) - \int_{\Gamma_2} qv d\gamma, \forall v \in V_0, \eta \in K. \tag{63}$$

Now,

$$a(\eta, v) = L_f(v), \forall v \in V_0, \eta \in K \tag{64}$$

and from the uniqueness of solution of the variational equality (4), we have:

$$\eta = u_f. \tag{65}$$

On the oder hand, from (62), (65) and the uniqueness of solution of the variational equality (22), it results that:

$$\xi = p_f$$

Now,

$$J_\alpha(g_{op_\alpha}) \leq J_\alpha(f^*), \forall f^* \in H$$

next,

$$J(f) = J_\alpha(f) \leq \liminf_{\alpha\to\infty} J_\alpha(g_{op_\alpha}) \leq \liminf_{\alpha\to\infty} J_\alpha(f^*) = \lim_{\alpha\to\infty} J_\alpha(f^*) = J(f^*)$$

and from the uniqueness of the optimal control we obtain that $f = g_{op}$. Therefore $\eta = u_f = u_{g_{op}}$ and $\xi = p_f = p_{g_{op}}$.

Moreover, from (61) and the following computation:

$$\begin{aligned}\lambda_1 \left\|p_{g_{op_\alpha}\alpha} - p_{g_{op}}\right\|_V^2 &\leq a_\alpha(p_{g_{op_\alpha}\alpha} - p_{g_{op}}, p_{g_{op_\alpha}\alpha} - p_{g_{op}}) \\ &= a_\alpha(p_{g_{op_\alpha}\alpha}, p_{g_{op_\alpha}\alpha} - p_{g_{op}}) - a(p_{g_{op}}, p_{g_{op_\alpha}\alpha} - p_{g_{op}}) \\ &= (u_{g_{op_\alpha}\alpha} - z_d, p_{g_{op_\alpha}\alpha} - p_{g_{op}}) - a(p_{g_{op}}, p_{g_{op_\alpha}\alpha} - p_{g_{op}})\end{aligned}$$

we have (54). From the optimality condition (24) it results that:

$$\|g_{op_\alpha} - g_{op}\|_H = \frac{1}{M} \left\|p_{g_{op}} - p_{g_{op_\alpha}\alpha}\right\|_H \leq \frac{1}{M} \left\|p_{g_{op}} - p_{g_{op_\alpha}\alpha}\right\|_V$$

and therefore (55) holds. Now, we have:

$$\begin{aligned}\lambda_1 \left\|u_{g_{op_\alpha}\alpha} - u_{g_{op}}\right\|_V^2 &\leq a_\alpha(u_{g_{op_\alpha}\alpha} - u_{g_{op}}, u_{g_{op_\alpha}\alpha} - u_{g_{op}}) \\ &= a_\alpha(u_{g_{op_\alpha}\alpha}, u_{g_{op_\alpha}\alpha} - u_{g_{op}}) - a_\alpha(u_{g_{op}}, u_{g_{op_\alpha}\alpha} - u_{g_{op}}) \\ &= L_{g_{op_\alpha}\alpha}(u_{g_{op_\alpha}\alpha} - u_{g_{op}}) - a(u_{g_{op}}, u_{g_{op_\alpha}\alpha} - u_{g_{op}}) - \alpha \int_{\Gamma_1} b(u_{g_{op_\alpha}\alpha} - b)\, d\gamma \\ &= a(u_{g_{op_\alpha}}, u_{g_{op_\alpha}\alpha} - u_{g_{op}}) - a(u_{g_{op}}, u_{g_{op_\alpha}\alpha} - u_{g_{op}}) \\ &= a(u_{g_{op_\alpha}} - u_{g_{op}}, u_{g_{op_\alpha}\alpha} - u_{g_{op}})\end{aligned}$$

and taking into account (55) and the fact that $u_{g_{op_\alpha}} \to u_{g_{op}}$ strongly in V when $\alpha \to \infty$ because (18), we get (56). ■

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