MAT

Serie A

Conferencias, seminarios y trabajos de Matemática

ISSN:1515-4904

4

VI Seminario sobre Problemas de Frontera Libre y sus Aplicaciones.

Segunda Parte

Departamento de Matemática, Rosario, Argentina
2001

UNIVERSIDAD AUSTRAL
FACULTAD DE CIENCIAS EMPRESARIALES

ISSN 1515-4904

MAT

SERIE A : CONFERENCIAS, SEMINARIOS Y TRABAJOS DE MATEMÁTICA

No. 4

VI SEMINARIO SOBRE PROBLEMAS DE FRONTERA LIBRE Y SUS APLICACIONES
Segunda Parte

Domingo A. Tarzia (Ed.)

INDICE

Rosario, Septiembre 2001

MAT – Serie A, 4 (2001), 33-38.

EXACT SOLUTIONS FOR PHASE CHANGE PROCESSES IN HUMID POROUS HALF SPACES

Eduardo A. SANTILLAN MARCUS (*) **and Domingo A. TARZIA** (*)(**)

(*) Departamento de Matemática, FCE, Univ. Austral,
Paraguay 1950, (S2000FZF) Rosario, Argentina.
(**) CONICET, Argentina.
E-mails: Eduardo.Santillan@fce.austral.edu.ar ; Domingo.Tarzia@fce.austral.edu.ar

Abstract. We review recent results concerning two different models for freezing (desublimation) in a humid porous half space with a heat flux condition of the type $q_0/\sqrt{t}$ at the fixed face $x = 0$. Exact solutions for the coupled temperature and moisture distributions are obtained in both models. In each case, an inequality for the positive coefficient q_0 is necessary and sufficient in order to obtain that explicit solution. Furthermore, an equivalence between a phase change problem with a temperature condition at $x = 0$ and the corresponding phase change process with a heat flux condition of that type on the fixed face for each model is also obtained.

Resumen. Se presenta una revisión de recientes resultados concernientes a dos modelos diferentes de congelamiento (desublimación) de un semi-espacio poroso húmedo con una dada condición de flujo del calor del tipo $q_0/\sqrt{t}$ en el borde fijo $x = 0$. Se obtienen soluciones exactas para las distribuciones de temperatura y humedad acopladas para ambos modelos. En cada caso, se deduce una desigualdad para el parámetro positivo q_0 que resulta ser una condición necesaria y suficiente para obtener las soluciones explícitas de tipo similaridad. Más aún, se demuestra también una equivalencia entre el problema de cambio de fase con una condición de temperatura constante en $x = 0$ y el correspondiente problema de cambio de fase con la dada condición de flujo de calor para ambos modelos.

Key Words: Phase change, porous medium, freezing, desublimation, evaporation, similarity method, moisture, exact solutions, free boundary problem, Stefan problem.

Palabras Claves: Cambio de fase, medio poroso, congelamiento, desublimación, evaporación, método de similaridad, húmedad, soluciones exactas, problema de frontera libre, problema de Stefan.

AMS Subject Classification: 35R35, 80A22, 35C05.

1.- INTRODUCTION

Heat and mass transfer with phase change problems, taking place in a porous medium, such as evaporation, condensation, freezing, melting, sublimation and desublimation, have wide application in separation processes, food technology, heat and mixture migration in soils and grounds (see [AlSo, Cr, Lun,Ta3]). Due to the non-linearity of the problem, solutions usually involve mathematical difficulties. Only a few exact solutions have been found for idealized cases [LaCl, CaJa, FPT, Li, Mi1, Mi2; Ta1, Ta2]. Mathematical formulation of the heat and mass transfer in capillary porous bodies has been established by Luikov [Lu1, Lu2, Lu3, LuMi]. Two differents models was presented in [Mi1] for solving the problem of evaporation of liquid moisture from a porous medium. For the problem of freezing (desublimation) of humid porous half-space, an exact solution is presented in [Mi2]. An exact solution for the desublimation problem in a porous medium with a temperature condition on the fixed face is also presented by [Li]. Other problems in this direction was given by [Ch, FGPR, FPT]. A large bibliography on free and moving boundary problems for the heat-diffusion equation was given in [Ta3].

In the following, two different analytical models of freezing (desublimation) of moisture in a humid porous medium with heat flux condition at $x=0$ of the type $q_0/\sqrt{t}$ (with $q_0>0$) will be presented. First, a desublimation process, following [Li, SaTa2] will be studied. An analytic model of this process will be defined and exact solutions for the coupled temperature and moisture distributions will be found. Furthermore, the position of the desublimation free boundary that divides the frozen and the humid vapour phases will be found too. An inequality for the coefficient q_0 is necessary and sufficient in order to obtain the corresponding explicit solution. Finally, an equivalence between a phase change problem with a constant temperature condition and a phase change process with a heat flux condition of the that type at the fixed face is also obtained. Secondly, a freezing process, following [Mi1, SaTa1] will be studied and similar conclusion of the first part can be obtained. This article generalizes the results obtained for the temperature distribution in the solid-liquid phase change problem due to [Ta1], for the desublimation problem (coupled of temperature and moisture distributions) in a porous humid half-space with a heat flux condition at the fixed face.

2.- Desublimation in humid porous half spaces

Let us consider a solid, rigid, porous half-space containing a uniform mixture of air and moisture in a vapor form. Initially the porous body is at a uniform moisture concentration C_i and an uniform temperature T_i. The vapor is desublimated by maintaining a heat flux condition of the type $q_0/\sqrt{t}$ on the surface at $x=0$. For the formulation of the problem, we made the same assumptions as made in [Li]:

(1) In the frozen region, $0<x<s(t)$, there is no moisture movement, where $s=s(t)$ locates the desublimation front. In the vapor region, $s(t)<x<+\infty$, there are heat and moisture mass flows;

(2) The convective terms in the vapor region are small and may be neglected;

(3) The thermophysical properties of the frozen and vapor regions remain respectively constants;

(4) The Soret effect, or the thermal diffusion, gives rise to a mass flux which is normally very small relative to the normal Fickian flux, and may be neglected.

The following differential equations describe the process of desublimation:

$$\frac{\partial T_1}{\partial t}=a_1\frac{\partial^2 T_1(x,t)}{\partial x^2},\quad 0<x<s(t),\quad t>0 \tag{1}$$

$$\frac{\partial T_2}{\partial t}=a_2\frac{\partial^2 T_2(x,t)}{\partial x^2},\qquad \frac{\partial C}{\partial t}=a_m\frac{\partial^2 C(x,t)}{\partial x^2},\quad s(t)<x<\infty\,,\ t>0 \tag{2}$$

where a_1 and a_2 are the volume averaged thermal diffusivities in the frozen and vapor regions, respectively, and a_m the volume averaged mass diffusivity of the vapor in the porous body. T_1 denotes the temperature in the frozen region, and T_2 and C denote the temperature and the moisture concentration of the vapor region respectively. The initial and boundary conditions and the heat and moisture mass balance at the desublimation front $x=s(t)$ can be described as follows:

$$T_2(\infty,t)=T_2(x,0)=T_i,\qquad C(\infty,t)=C(x,0)=C_i,\quad x>0,\quad t>0\ ;\quad s(0)=0 \tag{3}$$

$$k_1 \frac{\partial T_1}{\partial x}(0,t) = \frac{q_0}{\sqrt{t}} \ ,t > 0 \tag{4}$$

$$T_1(s(t),t) = T_2(s(t),t) = T_s \ , \qquad C(s(t),t) = C_s < C_i, \qquad t > 0 \tag{5}$$

$$k_1 \frac{\partial T_1}{\partial x}(s(t),t) - k_2 \frac{\partial T_2}{\partial x}(s(t),t) = C_0 L \dot{s}(t), \ a_m \frac{\partial C}{\partial x}(s(t),t) = (C_0 - C_s)\dot{s}(t), \ t > 0 \tag{6}$$

where k_1 and k_2 are the volume averaged thermal conductivities in the frozen and vapor regions respectively, L is the latent heat of desublimation, C_s is the moisture concentration of the vapor phase at the desublimation front and $C_0 (C_0 > C_s)$ is the moisture concentration of the frozen phase, which is an unknown of the problem and it must to be determined as a part of the solution. The heat flux condition (4) was firstly considered by [Ta1] for a solid-liquid phase-change problem. The solution of the desublimation process (1)-(6) is given by:

$$T_1(x,t) = T_s + \frac{q_0\sqrt{\pi a_1}}{k_1}\left(erf\left(x/2\sqrt{a_1 t}\right) - erf\left(\sqrt{a_2/a_1}\lambda\right)\right) \tag{7}$$

$$T_2(x,t) = T_i - \frac{T_i - T_s}{erfc(\lambda)} erfc\left(x/2\sqrt{a_2 t}\right), \quad C(x,t) = C_i - \frac{C_i - C_s}{erfc\left(\sqrt{a_2/a_1}\lambda\right)} erfc(x/2\sqrt{a_1 t}) \tag{8}$$

$$s(t) = 2\lambda\sqrt{a_2 t} \ , \qquad C_0 = C_s + \sqrt{\frac{a_m}{a_2}}\left(\frac{C_i - C_s}{\sqrt{\pi}\lambda}\right) F_1\left(\sqrt{\frac{a_m}{a_2}}\lambda\right) \tag{9}$$

where erf and $erfc$ are the error and the complimentary error functions, defined by

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-v^2)dv \ , \quad erfc(x) = 1 - erf(x) \ , \quad F_1(x) = \exp(-x^2)\Big/erfc(x) \tag{10}$$

and λ is a dimensionless constant which characterizes the desublimation front and must satisfy the following equation

$$q_0 \exp\left(-\frac{a_2}{a_1}\lambda^2\right) - \frac{k_2(T_i - T_s)}{\sqrt{\pi a_2}} F_1(\lambda) - \sqrt{\frac{a_m}{\pi}}(C_i - C_s) L F_1\left(\sqrt{\frac{a_2}{a_m}}\lambda\right) = C_s L \sqrt{a_2}\lambda \ , \lambda > 0 \,. \tag{11}$$

Theorem 1.We have: (i) If the coefficient q_0 verifies the condition

$$q_0 > \frac{k_2(T_i - T_s)}{\sqrt{\pi a_2}} + (C_i - C_s)L\sqrt{\frac{a_m}{\pi}} \ , \tag{12}$$

then there exists one and only one solution $\lambda > 0$ of the equation (11). In this case, the solution of the problem (1)-(6) is given by (7)-(9) for that λ.

(ii) If $q_0 \le \frac{k_2(T_i - T_s)}{\sqrt{\pi a_2}} + (C_i - C_s)L\sqrt{\frac{a_m}{\pi}}$, then there is no solution of the problem (1)-(6) as a phase change problem; it is only a heat conduction problem for the initial vapor phase.

Proof. See [SaTa2]. ■

We assume the case when data verify condition (12). Then we can compute $T_1(0,t)$ wich is given by

$$T_0 = T_1(0,t) = T_s - \frac{q_0\sqrt{\pi a_1}}{k_1} erf\left(\sqrt{\frac{a_2}{a_1}}\lambda\right) < T_s \tag{13}$$

Next, we can enunciate the following property:

Lemma 2. A phase change problem (1)-(6) for temperature and moisture distributions in a porous half-space with a heat flux condition on the surface $x = 0$ verifying the condition (12), is equivalent to a phase change problem (1)-(3), (5)-(6) with a temperature condition (14) at $x = 0$. Moreover, the relationship among q_0 and T_0 is given by (13), where λ is the coefficient which characterizes the free boundary given by the unique solution of the equation (11). ■

3.- Freezing of humid porous half-space

Let us consider the flow of heat and moisture through a porous half-space during freezing [Mi2]. The position of phase change front at time t is given by $x = s(t)$. It divides the porous body into two regions. In the freezing region, $0 < x < s(t)$, there is no moisture movement and the temperature distribution t_1 is described by the heat equation

$$\frac{\partial t_1}{\partial t} = a_1 \frac{\partial^2 t_1(x,t)}{\partial x^2}, \quad 0 < x < s(t), \quad t > 0 \tag{14}$$

The region $s(\tau) < x < +\infty$ is humid capillary porous body in which there are coupled heat and moisture flows. The process is described by the well known Luikov's system [LuMi] for the case $\varepsilon = 0$ (ε is the phase conversion factor of liquid into vapor):

$$\frac{\partial t_2}{\partial t} = a_2 \frac{\partial^2 t_2(x,t)}{\partial x^2}, \quad \frac{\partial u}{\partial t} = a_m \frac{\partial^2 u}{\partial x^2} + a_m \delta \frac{\partial^2 t_2}{\partial x^2}, \qquad x > s(t), \quad t > 0 \tag{15}$$

The initial distributions of temperature and moisture are uniform:

$$t_2(x,0) = t_2(+\infty,0) = t_0\ ;\ u(x,0) = u(+\infty,t) = u_0\ ;\ x > 0, t > 0 \tag{16}$$

It is assumed that on the surface of the half-space the heat flux depends on the time in the way (4), like in [Ta1],where $q_0 > 0$ is a coefficient which characterizes the heat flux at the fixed face $x = 0$. On the freezing front, there exists an equality between the temperatures:

$$t_1(s(t),t) = t_2(s(t),t) = t_v, \quad t > 0 \tag{17}$$

where $t_v < t_0$. Heat and moisture balance at the freezing front yields

$$k_1 \frac{\partial t_1}{\partial x}(s(t),t) - k_2 \frac{\partial t_2}{\partial x}(s(t),t) = u(s(t),t)\rho_2 r \dot{s}(t), \quad t > 0 \tag{18}$$

$$\frac{\partial u}{\partial x}(s(t),t) + \frac{\partial t_2}{\partial x}(s(t),t) = 0, \quad t > 0 \tag{19}$$

If we consider the next transformations (l_0 is a characteristic length):

$$X = x/l_0; \qquad F_0 = a_2 t / l_0^2 \ (\textit{Fourier number}); \qquad S(F_0) = s(t)/l_0$$
$$T_i(X,F_0) = (t_i(x,t) - t_v)/(t_0 - t_v);\ i = 1,2 \quad ; \qquad \Theta(X,F_0) = (u_0 - u(x,t))/u_0 \tag{20}$$

after some computations we obtain the following solution:

$$T_1(X,F_0) = [\sqrt{\pi\, a_1}\, q_0 / k_1(t_0 - t_v)]\left(erf\left(X/2\sqrt{a_{12}F_0}\right) - erf\left(\lambda/\sqrt{a_{12}}\right)\right) \tag{21}$$

$$T_2(X,F_0) = \left(1 - erf(\lambda)\right)^{-1}\left(erf\left(X/2\sqrt{F_0}\right) - erf(\lambda)\right), \qquad S(F_0) = 2\lambda\sqrt{F_0} \tag{22}$$

$$\Theta(X,F_0) = 1 - [\exp(\lambda^2/L_u)/(\sqrt{L_u}\ \exp(\lambda^2) erfc(\lambda))] erfc(X/2\sqrt{L_u F_0}) \tag{23}$$

where the parameter λ must be determined as the solution of the following equation:

$$\left\{\frac{\sqrt{\pi a_2}\, q_0}{k_2(t_0 - t_v)}\right\}\exp\left(\frac{-\lambda^2}{a_{12}}\right) - \frac{\exp(-\lambda^2)}{1 - erf\lambda} = \sqrt{\pi} K_0 \lambda \left(1 - \left[\frac{1 - L_u}{L_u P_n}\right](1 - H(\lambda))\right), \quad \lambda > 0 \tag{24}$$

where L_u, P_n and K_0 are the Kuikov, Posnov and Kossovitch numbres, $a_{12} = a_1/a_2$ and the real functions Q and H are defined by:

$$Q(x) = \sqrt{\pi}\, x \exp(-x^2)(1 - erf(x)) \quad ; \quad H(x) = \frac{Q(x/L_u)}{Q(x)} \tag{25}$$

We can enunciate the following property:

Theorem 3: We have:(i) If

$$q_0 > \frac{k_2(t_0 - t_v)}{\sqrt{\pi\, a_2}} \equiv q_{00} \tag{26}$$

then there exists one and only one solution $\lambda > 0$ of the equation (24).

(ii) If $q_0 \le q_{00}$ then there is no solution of the problem (14)-(19) as a phase change problem; it is only a heat conduction problem for the initial phase. Moreover, the limit case $q_0 = q_{00}$ can be interpretated as the limit of the solution for the case $q_0 > q_{00}$ when $r \to \infty$.

Proof. See [SaTa1]. ∎

If we consider the case where data verify condition (26) then we can compute the temperature at the fixed $x = 0$ which is given by:

$$t_s = t_1(0,t) = t_v - \sqrt{\pi\, a_1}\,\frac{q_0}{k_1} erf\left(\frac{\lambda}{\sqrt{a_{12}}}\right) < t_v\ . \tag{27}$$

Next, we can enunciate the following property:

Lemma 4. A freezing problem for temperature and moisture distributions in a porous half-space with a heat flux condition on the surface $x = 0$ of the type (4), is equivalent to a phase change problem with a temperature condition at $x = 0$ considering $t_1(0,\tau) = t_s < t_v$. Moreover, the relationship among q_0 and t_s is given by (27) where λ is the coefficient which characterizes the free boundary which is the unique solution of a certain equation. ■

Remark 1. An inequality for the coefficient λ which characterizes the phase-change interphase can be obtained for both problems (in sections 2 and 3) when a constant temperature boundary condition is impossed at $x = 0$ [SaTa1, SaTa2].

References

[AlSo] Alexiades, V. - Solomon, A. D., "*Mathematical modeling of melting and freezing processes*", Hemisphere – Taylor & Francis, Washington (1993).
[CaJa] Carslaw, H. S. - Jaeger, J. C., "*Conduction of heat in solids*", Clarendon Press, Oxford (1959)
[Ch] Cho, S. H., "*An exact solution of the coupled phase change problem in a porous medium*", Int. J. Heat and Mass Transfer, 18, (1975), 1139-1142.
[Cr] Crank, J , "*Free and moving boundary problems*", Clarendon Press, Oxford (1984).
[FGPR] Fasano, A. - Guan, Z. - Primicerio, M. - Rubinstein, I., "*Thawing in saturated porous media*", Meccanica, 28 (1993), 103-109.
[FPT] Fasano, A. - Primicerio, M. - Tarzia, D. A., "*Similarity solutions in class of thawing processes*", Math. Models Methods Appl. Sci., 9(1999), 1-10.
[LaCl] Lamé, G. - Clapeyron, B. P., "*Memoire sur la soldification par refroidissement d'un globe liquide*", Annales Chimie Physique, 47 (1831), 250-256.
[Li] Lin, S , "*An exact solution of the desublimation problem in a porous medium*", Int. J. Heat and Mass Transfer, 25, (1982), 625-630.
[Lu1] Luikov, A. V., "*Heat and mass transfer in capillary-porous bodies*", Pergamon, Oxford (1966).
[Lu2] Luikov, A. V., "*Systems of differential equations of heat and mass transfer in capillary porous bodies*", Int. J. Heat Mass Transfer, 18 (1975), 1-14.
[Lu3] Luikov, A. V., "*Heat and mass transfer*", MIR Publishers, Moscow (1978).
[LuMi] Luikov, A. V. – Mikhailov Yu A., "*Theory of energy and mass transfer*", Pergamon, New York (1968).
[Lun] Lunardini, V. J. , "*Heat transfer with freezing and thawing*", Elsevier, Amsterdam (1991).
[Mi1] Mikhailov, M. D., "*Exact solution of temperature and moisture distributions in a porous half-space with moving evaporation front*", Int. J. Heat Mass Transfer, 18 (1975), 797-804.
[Mi2] Mikhailov, M. D., "*Exact solution for freezing of humid porous half-space*", Int. J. Heat Mass Transfer, 19 (1976), 651-655.
[SaTa1] Santillan Marcus, E. A. and Tarzia, D. A., "*Explicit solution for freezing of humid porous half-space with a heat flux condition*", Int. J. Eng. Sci., 38 (2000), 1651-1665.
[SaTa2] Santillan Marcus, E. A. and Tarzia, D. A., "*Explicit solutions for the desublimation problem in a humid porous half-space with a heat flux condition*", Información Tecnológica, 11 # 4 (2000), 19-24.
[Ta1] Tarzia, D. A., "*An inequality for the coefficient* σ *of the free boundary* $s(t) = 2\sigma\sqrt{t}$ *of the Neumann solution for the two-phase Stefan problem*", Quart. Appl. Math., 39 (1981), 491-497.
[Ta2] Tarzia, D. A., "*Soluciones exactas del problema de Stefan unidimensional*", Cuadern. Inst. Mat. B. Levi, 12 (1985), 5 - 36.
[Ta3] Tarzia D. A., "*A bibliography on moving free-boundary problems for the heat diffusion equation. The Stefan and related problems*", MAT-SerieA, # 2 (2000), (with 5869 titles on the subject, 300 pages). See also www.austral.edu.ar/MAT-SerieA/2(2000)/.